Strength of Materials: An Undergraduate Text

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Chapter Solutions

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Chapter 1 Solutions

1.1 The cross-sectional area, *A*, of the bar is

$$A = \pi\left(50\text{x}10^{-3}\right)^2 = 7.85\text{x}10^{-3}\,\text{m}^2$$

The axial stress, σ, due to the axial load *P*=30kN is

$$\sigma = \frac{P}{A} = \frac{30\text{x}10^3}{7.85\text{x}10^{-3}} = 3.82\text{MPa}$$

1.2 From (1.6) the axial strain, *e*, is

$$\varepsilon = \frac{\sigma}{E} = \frac{3.82\text{x}10^6}{210\text{x}10^9} = 18.2\text{x}10^{-6}$$

From (1.3) the axial extension, δ, is

$$\delta = \varepsilon L = 18.2\text{x}10^{-6}\,\text{x}0.5 = 9.1\text{x}10^{-6}\,\text{m}$$

1.3 From (1.16) the in-plane strains are

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \quad , \quad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right]$$

Re-arranging the first equation for σ_{xx}

$$\sigma_{xx} = E\varepsilon_{xx} + \nu\sigma_{yy}$$

and substituting into the second equation then σ_{yy} is found to be

$$\sigma_{yy} = \frac{E}{1-\nu^2}\left[\varepsilon_{yy} + \nu\varepsilon_{xx}\right] = \frac{70\text{x}10^9}{1-0.28^2}\left[60 + 0.28\text{x}40\right]\text{x}10^{-6} = 5.4\text{MPa}$$

Substituting σ_{yy} into ε_{xx} and re-arranging for σ_{xx}

$$\sigma_{xx} = \frac{E}{1-\nu^2}\left[\varepsilon_{xx} + \nu\varepsilon_{yy}\right] = \frac{70\text{x}10^9}{1-0.28^2}\left[40 + 0.28\text{x}60\right]\text{x}10^{-6} = 4.3\text{MPa}$$

1.4 From (1.23) the bulk modulus is

$$K = \frac{E}{3(1-2\nu)} = \frac{210\text{x}10^9}{3(1-2\text{x}0.3)} = 175\text{GPa}$$

1.5 With *E*=σ/ε, σ=*P*/*A*, ε=δ/*L* and δ=α(Δ*T*) then *P* is given by

$$P = \frac{EA\alpha(\Delta T)}{L} = \frac{120\text{x}10^9\,\text{x}7.1\text{x}10^{-4}\,\text{x}11\text{x}10^{-6}(100)}{0.75} = 125\text{kN}$$

illustrating that a compressive axial load of 125kN is required to cancel out the extension due to thermal expansion.

1.6 The compressive stresses of each bar are

$$\sigma_{copper} = \frac{P}{A_{copper}} = \frac{150\text{x}10^3}{\pi\left(25\text{x}10^{-3}\right)^2} = 76\text{MPa}$$

$$\sigma_{steel} = \frac{P}{A_{steel}} = \frac{150\text{x}10^3}{\pi\left(37.5\text{x}10^{-3}\right)^2} = 34\text{MPa}$$

The contractions of each bar are

$$\delta_{copper} = \frac{PL_{copper}}{E_{copper}A_{copper}} = \frac{150\text{x}10^3\text{x}0.5}{120\text{x}10^9\text{x}\pi\left(25\text{x}10^{-3}\right)^2} = 318\text{x}10^{-6}\,\text{m}$$

$$\delta_{steel} = \frac{PL_{steel}}{E_{steel}A_{steel}} = \frac{150\text{x}10^3\text{x}0.6}{200\text{x}10^9\text{x}\pi\left(37.5x10^{-3}\right)^2} = 102\text{x}10^{-6}\,\text{m}$$

The total contraction of the composite bar is $\delta=\delta_{copper}+\delta_{steel}=420\text{x}10^{-6}$m.

1.7 From (1.31,1.32,1.36) the in-plane strains ε_{xx}, ε_{yy} and γ_{xy} are given by

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \quad , \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = \frac{1}{E}\left(\sigma_{yy} - \nu\sigma_{xx}\right)$$

$$\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 0$$

Chapter 2 Solutions

2.1 Figure Sol2.1 illustrates a circle of radius R with an elemental strip of thickness dy at distance y from the x-axis. The area, dA, of the elemental strip is therefore

$$dA = 2xdy = 2\sqrt{R^2 - y^2}\,dy$$

The area of the strip is therefore given by, (2.1)

$$A = \iint_A dA = 2\int_{-R}^{+R} \sqrt{R^2 - y^2}\,dy$$

The integration is assisted by making the substitution y=Rsinθ, dy=Rcosθdθ

$$A = 4R^2 \int_0^{p/2} \cos^2 dJ \;= 4R^2 \int_0^{p/2} \left[\frac{1}{2}(1 + \cos 2J)\right] dJ \;= 4R^2 \left[\frac{J}{2} + \frac{\sin 2J}{2}\right]_0^{p/2} pR^2$$

The first moment of area Q_x is, (2.2)

$$Q_x = \iint_A y dA = \int_{-R}^{+R} y\left(2\sqrt{R^2 - y^2}\,dy\right) = 2\int_{-R}^{+R} y\sqrt{R^2 - y^2}\,dy = -\frac{2}{3}\left[\left(R^2 - y^2\right)^{3/2}\right]_{-R}^{+R} = 0$$

Similarly, it is found that Q_y=0. Therefore, as expected the coordinates of the centroid are x_c=Q_y/A=0 and y_c=Q_x/A=0.

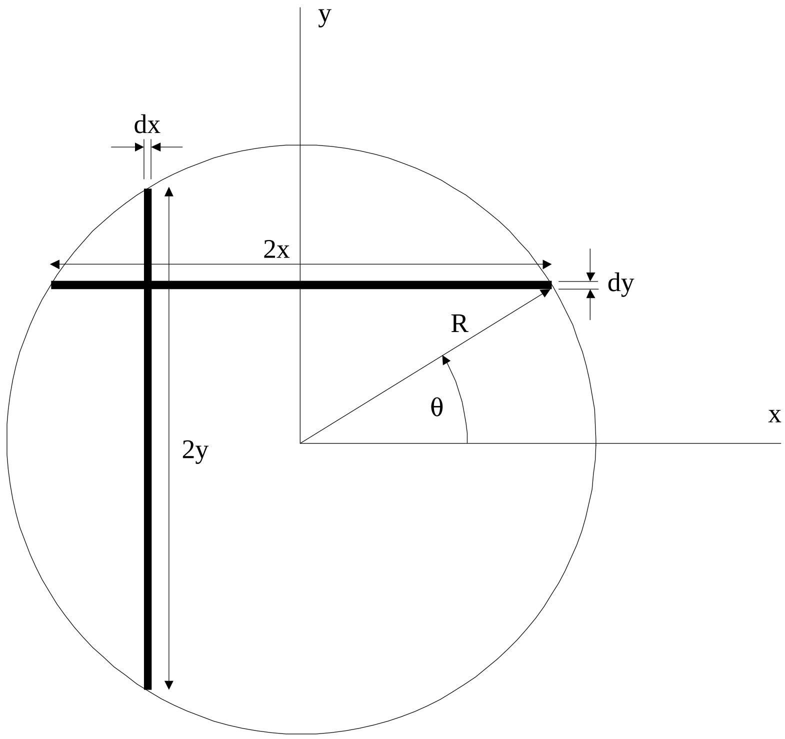


Figure Sol2.1. A circle of radius R and elemental strip dy.

2.2 The derivation of the area, first moments of area and centroid are analogous to those outlined in Exercise 2.1 except that the range of integration is now [0:R]. Thus, the area of the semicircle is

$$A = \iint_A dA = 2\int_0^R \sqrt{R^2 - y^2}\,dy = 2R^2 \int_0^{p/2} \cos^2 J dJ \;= 2R^2 \left[\frac{J}{2} + \frac{\sin 2J}{2}\right]_0^{p/2} = \frac{pR^2}{2}$$

The first moment of area Q_x is, using the elemental strip dy shown in Figure Sol2.1

$$Q_x = \iint_A dA = 2\int_0^R y\sqrt{R^2-y^2}dy = -\frac{2}{3}\left[\left(R^2-y^2\right)^{3/2}\right]_0^R = \frac{2R^3}{3}$$

The y-coordinate of the centroid is therefore

$$y_c = \frac{Q_x}{A} = \frac{2R^3/3}{pR^2/2} = \frac{4R}{3p}$$

The first moment of area Q_y is, using the elemental strip dx shown in Figure Sol2.1 in which dA=ydx

$$Q_y = \iint_A x dA = \int_{-R}^{+R} x\sqrt{R^2-x^2}dx = -\frac{1}{3}\left[\left(R^2-x^2\right)^{3/2}\right]_{-R}^{+R} = 0$$

and hence $x_c=Q_y/A=0$.

2.3 Figure Sol2.3 illustrates an ellipse with elemental strip of length 2x and width dy. The area A of the ellipse is therefore, noting that the equation of an ellipse is $x^2/a^2+y^2/b^2=1$

$$A = \iint_A dA = \int_{-b}^{b} 2x dy = 2a\int_{-b}^{b}\sqrt{1-\frac{y^2}{b^2}}dy = \frac{2a}{b}\int_{-b}^{b}\sqrt{b^2-y^2}dy$$

Using the standard indefinite integral

$$\int\sqrt{a^2-x^2}dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)+\frac{1}{2}x\sqrt{a^2-x^2}$$

then the area is found to be

$$A = \frac{2a}{b}\left[\frac{b^2}{2}\sin^{-1}\left(\frac{y}{b}\right)+\frac{1}{2}y\sqrt{b^2-y^2}\right]_{-b}^{b} =$$

$$= \frac{2a}{b}\left\{\left[\frac{b^2}{2}\sin^{-1}(1)\right]-\left[\frac{b^2}{2}\sin^{-1}(-1)\right]\right\} = \frac{2a}{b}\left[\frac{pb^2}{4}+\frac{pb^2}{4}\right] = pab$$

The second moment of area I_x is, with dA=2xdy

$$I_x = \iint_A y^2 dA = \int_{-b}^{b} y^2 2x dy = \int_{-b}^{b} y^2 2a\sqrt{1-\frac{y^2}{b^2}}dy = 2a\int_{-b}^{b} y^2\sqrt{1-\frac{y^2}{b^2}}dy$$

Using the standard indefinite integral

$$\int x^2\sqrt{ax^2+c}dx = \frac{x}{4a}\sqrt{\left(ax^2+c\right)^3} - \frac{cx}{8a}\sqrt{ax^2+c} - \frac{c^2}{8a\sqrt{-a}}\sin^{-1}\left(x\sqrt{\frac{-a}{c}}\right) \quad ; \quad a<0$$

then I_x is

$$I_x = 2a\left\{\frac{y}{-\left(4/b^2\right)}\sqrt{\left(1-\frac{y^2}{b^2}\right)^3} - \frac{y}{8\left(-1/b^2\right)}\sqrt{1-\frac{y^2}{b^2}} - \frac{1}{\left(-8/b^2\right)\sqrt{1/b^2}}\sin^{-1}\left(y\sqrt{\frac{1/b^2}{1}}\right)\right\}_{-b}^{b} = \frac{pab^3}{4}$$

Similarly, it can be shown that $I_y=\pi a^3 b/4$.

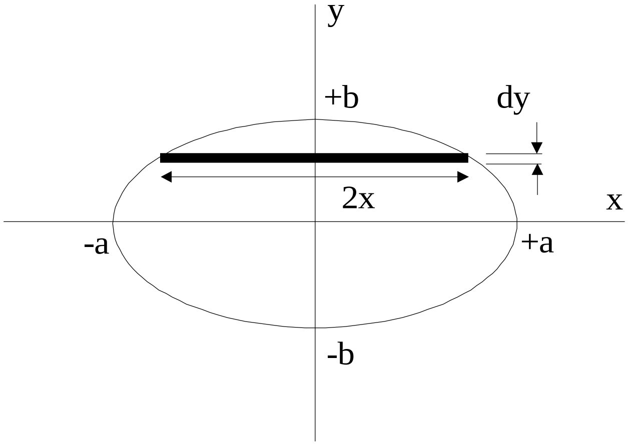


Figure Sol2.3. An ellipse with half major and minor axes *a* and *b*.

2.4 The centroidal cordinates for rectangles 1 and 2 are (x_{c1},y_{c1})=(27.5,2.5) and (x_{c2},y_{c2})=(2.5,25). The areas of rectangles 1 and 2 are A_1=225mm^2 and A_2=250mm^2 with the total cross-sectional area equal to $A=A_1+A_2$=475mm^2. From (2.2) and (2.3) Q_y and x_c are

$$Q_y = \sum_i x_{ci} A_i = x_{c1} A_1 + x_{c2} A_2 = 6{,}812\text{mm}^3 \quad , \quad x_c = \frac{Q_y}{A} = 14.34\text{mm}$$

and similarly with Q_x and y_c given by

$$Q_x = \sum_i y_{ci} A_i = y_{c1} A_1 + y_{c2} A_2 = 6{,}812\text{mm}^3 \quad , \quad y_c = \frac{Q_x}{A} = 14.34\text{mm}$$

with $x_c=y_c$ and $Q_x=Q_y$ due to the symmetry of the bracket about the (x,y) axes.

From (2.5) and (2.6) the second moments of area of rectangles 1 and 2 are given by, with respect to the centroidal axes

$$I_{xc1} = \frac{45(5)^3}{12} = 468\text{mm}^4 \quad , \quad I_{yc1} = \frac{5(45)^3}{12} = 37{,}968\text{mm}^4$$

$$I_{xc2} = \frac{5(50)^3}{12} = 52{,}083\text{mm}^4 \quad , \quad I_{yc2} = \frac{50(5)^3}{12} = 520\text{mm}^4$$

Use of the parallel-axis theorem (2.8) and (2.9) for rectangle 1 gives I_x and I_y with respect to axes (x,y)

$$I_{x1} = I_{xc1} + A_1 y_{c1}^2 = 1{,}875\text{mm}^4 \quad , \quad I_{y1} = I_{yc1} + A_1 x_{c1}^2 = 208{,}125\text{mm}^4$$

and similarly for rectangle 2

$$I_{x2} = I_{xc2} + A_2 y_{c2}^2 = 208{,}333\text{mm}^4 \quad , \quad I_{y2} = I_{yc2} + A_2 x_{c2}^2 = 2{,}083\text{mm}^4$$

Finally, I_x and I_y are given by

$$I_x = I_{x1} + I_{x2} = 210{,}208\text{mm}^4 \quad , \quad I_y = I_{y1} + I_{y2} = 210{,}208\text{mm}^4$$

with $I_x=I_y$ as expected.

2.5 From (2.10) with dA=$b$dy then I_{xy} is given by

$$I_{xy} = \iint_A xydA = bx\int_{-h/2}^{h/2} ydy = bx\left[\frac{y^2}{2}\right]_{-h/2}^{h/2} = 0$$

and is seen to be equal to zero due to the symmetry about the coordinate axes.

2.6 From (2.13) the polar second moment of area for the ellipse of Exercise 2.3 is

$$I_p = I_x + I_y = \frac{pa^3 b}{4} + \frac{pa^3 b}{4} = \frac{pab}{4}\left(a^2 + b^2\right)$$

2.7 From (2.15) the radii of gyration r_x and r_y for the ellipse of Exercise 2.3 are

$$r_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{pab^3 / 4}{pab}} = \frac{b}{2} \quad , \quad r_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{pa^3 b / 4}{pab}} = \frac{a}{2}$$

Chapter 3 Solutions

3.1 From (3.12) the maximum shear stress is given by

$$t_{\max} = \frac{16T}{pD^3} = \frac{16(10\text{x}10^3)}{p\left(50\text{x}10^{-3}\right)^3} = 407\text{MPa}$$

3.2 From (3.1) the polar moment of area, J, is

$$J = \frac{pD^4}{32} = \frac{p\left(50\text{x}10^{-3}\right)^4}{32} = 61.36\text{x}10^{-6}\,\text{m}^4$$

and from (3.11) the angle of twist is

$$q = \frac{TL}{GJ} = \frac{10\text{x}10^3\text{x}1.25}{80\text{x}10^9\left(61.36\text{x}10^{-6}\right)} = 2.55\text{x}10^{-3}\text{radians} = 0.15^\circ$$

3.3 From Example 3.1 then the maximum shear stress occurs at the smallest diameter of d_1=50mm

$$t_{\max} = \frac{16T}{pd_1^3} = \frac{16\left(12\text{x}10^3\right)}{p\left(50\text{x}10^{-3}\right)^3} = 489\text{MPa}$$

and the angle of twist is

$$q = \frac{32TL}{3pG\left(d_2 - d_1\right)}\left[\frac{1}{d_1^3} - \frac{1}{d_2^3}\right]$$

$$= \frac{32\left(12\text{x}10^3\right)\text{x}1}{3p\text{x}80\text{x}10^9\left(75\text{x}10^{-3} - 50\text{x}10^{-3}\right)}\left[\frac{1}{\left(50\text{x}10^{-3}\right)^3} - \frac{1}{\left(75\text{x}10^{-3}\right)^3}\right] = 0.1147\text{radians} = 6.57^\circ$$

3.4 From (3.16) the mean shear stress is

$$t_m = \frac{T}{2pR_m^2 t} = \frac{100}{2p\left(31.25\text{x}10^{-3}\right)^2\left(0.1\text{x}10^{-3}\right)} = 163\text{MPa}$$

3.5 Re-arranging (3.16) for wall-thickness t

$$t = \frac{T}{2pR_m^2 t_m} = \frac{1.5\text{x}10^3}{2p\left(40\text{x}10^{-3}\right)^2\left(65\text{x}10^6\right)} = 2.3\text{mm}$$

3.6 From (3.1) and (3.2) the polar moments of area of the inner solid bar, J_A, and outer tube, J_B, are

$$J_A = \frac{pR_A^4}{2} = \frac{p\left(12.5\text{x}10^{-3}\right)^4}{2} = 3.83\text{x}10^{-8}\,\text{m}^4$$

$$J_B = \frac{p}{2}\left(R_B^4 - R_A^4\right) = \frac{p}{2}\left[\left(25\text{x}10^{-3}\right)^4 - \left(12.5\text{x}10^{-3}\right)^4\right] = 5.75\text{x}10^{-7}\,\text{m}^4$$

From (3.21) the angle of twist is

$$\theta = \frac{TL}{G_A J_A + G_B J_B} = \frac{5\text{x}10^3(0.75)}{45\text{x}10^9\left(3.83\text{x}10^{-8}\right) + 30\text{x}10^9\left(5.75\text{x}10^{-7}\right)} = 0.1976\text{radians} = 11.324^\circ$$

3.7 From (3.20) the torques in the inner solid bar, T_A, and outer tube, T_B, are

$$T_A = \left(\frac{G_A J_A}{G_A J_A + G_B J_B}\right)T = \left(\frac{46\text{x}10^9\left(3.83\text{x}10^{-8}\right)}{46\text{x}10^9\left(3.83\text{x}10^{-8}\right) + 30\text{x}10^9\left(5.75\text{x}10^{-7}\right)}\right)5\text{x}10^3 = 454\text{Nm}$$

$$T_B = \left(\frac{G_B J_B}{G_A J_A + G_B J_B}\right)T = \left(\frac{30\text{x}10^9\left(5.75\text{x}10^{-7}\right)}{46\text{x}10^9\left(3.83\text{x}10^{-8}\right) + 30\text{x}10^9\left(5.75\text{x}10^{-7}\right)}\right)5\text{x}10^3 = 4.55\text{kNm}$$

and from (3.22) the corresponding maximum shear stresses are

$$\tau_{A,\max} = \frac{T_A R_A}{J_A} = \frac{454\left(12.5\text{x}10^{-3}\right)}{3.83\text{x}10^{-8}} = 148\text{MPa}$$

$$\tau_{B,\max} = \frac{T_B R_B}{J_B} = \frac{4.55\text{x}10^3\left(25\text{x}10^{-3}\right)}{5.75\text{x}10^{-7}} = 198\text{MPa}$$

Chapter 4 Solutions

4.1 From (4.2) and (4.4) the principal stresses in the pipe are

$$\sigma_1 = \frac{pr}{t} = \frac{50\text{x}6.895\text{x}10^3\text{x}0.8}{15\text{x}10^{-3}} = 18.4\text{MPa} \quad , \quad \sigma_2 = \frac{\sigma_1}{2} = 9.2\text{MPa}$$

4.2 From the Hookian equations (1.16) we have

$$\varepsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\sigma_{\theta\theta}\right] \quad , \quad \varepsilon_{\theta\theta} = \frac{1}{E}\left[\sigma_{\theta\theta} - \nu\sigma_{zz}\right]$$

Solving these for the in-plane stresses then we have

$$\sigma_{zz} = \frac{E}{1-\nu^2}\left[\varepsilon_{zz} + \nu\varepsilon_{\theta\theta}\right] = \frac{70\text{x}10^9}{1-0.3^2}\left[429 + 0.3\text{x}1{,}821\right]\text{x}10^{-6} = 75\text{MPa}$$

$$\sigma_{\theta\theta} = \frac{E}{1-\nu^2}\left[\varepsilon_{\theta\theta} + \nu\varepsilon_{zz}\right] = \frac{70\text{x}10^9}{1-0.3^2}\left[1821 + 0.3\text{x}429\right]\text{x}10^{-6} = 150\text{MPa}$$

The maximum in-plane shear stress is, (4.7)

$$\tau_{\max} = \frac{\sigma_1 - \sigma_2}{2} = \frac{\sigma_{\theta\theta} - \sigma_{zz}}{2} = \frac{(150-75)\text{x}10^6}{2} = 37.5\text{MPa}$$

4.3 With the pressure p equal to ρgh and $\sigma_{\theta\theta}=\sigma_2$ not exceeding the maximum allowable stress σ_{allow}=300MPa/S where S(=10) is the safety factor then from (4.11) we find the maximum permissible depth of water to be

$$h = \frac{2t\sigma_{allow}}{\rho grS} = \frac{2\left(25\text{x}10^{-3}\right)300\text{x}10^6}{1000\text{x}9.81\text{x}1\text{x}10} = 153\text{m}$$

4.4 From (4.5) and (4.12) the circumferential strains for the cylinder, ε_c, and hemi-spherical end caps, ε_s, are

$$\varepsilon_c = \frac{pr}{Et_c}\left(1 - \frac{\nu}{2}\right) \quad , \quad \varepsilon_s = \frac{pr}{2Et_s}\left(1-\nu\right)$$

where E and ν denote Young's modulus and Poisson's ratio respectively, p is the internal pressure, r is the radius of the vessel and t_c and t_s are the wall thicknesses of the cylinder and hemi-spherical ends respectively. Equivalence of the circumferential strains ε_c and ε_s yields

$$t_s = \left(\frac{1-\nu}{2-\nu}\right)t_c$$

and with t_c=1mm then t_s=0.4mm.

4.5 From (4.9) the absolute maximum shear is equal to

$$\tau_{\max} = \frac{\sigma_1}{2} = \sigma_2 = \frac{pr}{2t}$$

The cylinder pressure, p, is

$$p = \frac{P}{A} = \frac{P}{\pi r^2}$$

where P is the force acting on the piston and r is the piston radius. Substituting p into τ_{max} then

$$\tau_{max} = \frac{\tau_Y}{S} = \frac{P}{2\pi r t}$$

where τ_Y is the yield stress in pure shear and S is the safety factor. Re-arranging for the cylinder wall thickness t

$$t = \frac{SP}{2\pi r \tau_Y} = \frac{2(50\text{x}10^3)}{2\pi(40\text{x}10^{-3})150\text{x}10^6} = 2.65\text{mm}$$

4.6 Re-arranging (4.11) for internal pressure p with σ_{UTS} denoting the ultimate tensile stress then

$$p = \frac{2\sigma_{UTS} t}{r} = \frac{2\text{x}737\text{x}10^6(10^{-3})}{65\text{x}10^{-3}} = 227\text{MPa}$$

4.7 From (4.19) I_x and I_y are given by

$$I_x = I_y = \pi r^3 t = \pi\text{x}25^3\text{x}1 = 49{,}087\text{mm}^4$$

Chapter 5 Solutions

5.1 From the equations of equilibrium the two unknown reactions R_A and R_B are found

$$\sum F^v = 0: \quad R_A + R_B - 30(2) - 40 = 0 \quad \Rightarrow \quad R_A + R_B = 100\text{kN}$$

$$\sum M = 0: \quad R_B(7) - 60(4) - 40(8) = 0 \quad \Rightarrow \quad R_A = 80\text{kN} \quad \text{and} \therefore R_A = 20\text{kN}$$

The shear force, V_x, and bending moment, M_x, for the four intervals i) $0<x<3$, ii) $3<x<5$, iii) $5<x<7$ and iv) $7<x<8$ are:

i) **$0<x<3$**

$$V_x = R_A = 20\text{kN}$$

$$M_x = R_A x = 20x\text{kNm}$$

ii) **$3<x<5$**

$$V_x = R_A - 30(x-3) = 20 - 30(x-3)\text{kN}$$

$$M_x = R_A x - 30\frac{(x-3)^2}{2} = 20x - 30\frac{(x-3)^2}{2}\text{kNm}$$

iii) **$5<x<7$**

$$V_x = R_A - 60 = -20\text{kN}$$

$$M_x = R_A x - 60(x-4) = 20x - 60(x-4)\text{kNm}$$

iv) **$7<x<8$**

$$V_x = R_A + R_B - 60 = 40\text{kN}$$

$$M_x = R_A x + R_B(x-7) - 6(x-4) = 20x + 80(x-7) - 6(x-4)\text{kNm}$$

with V_x and M_x illustrated in Figure Sol5.1.

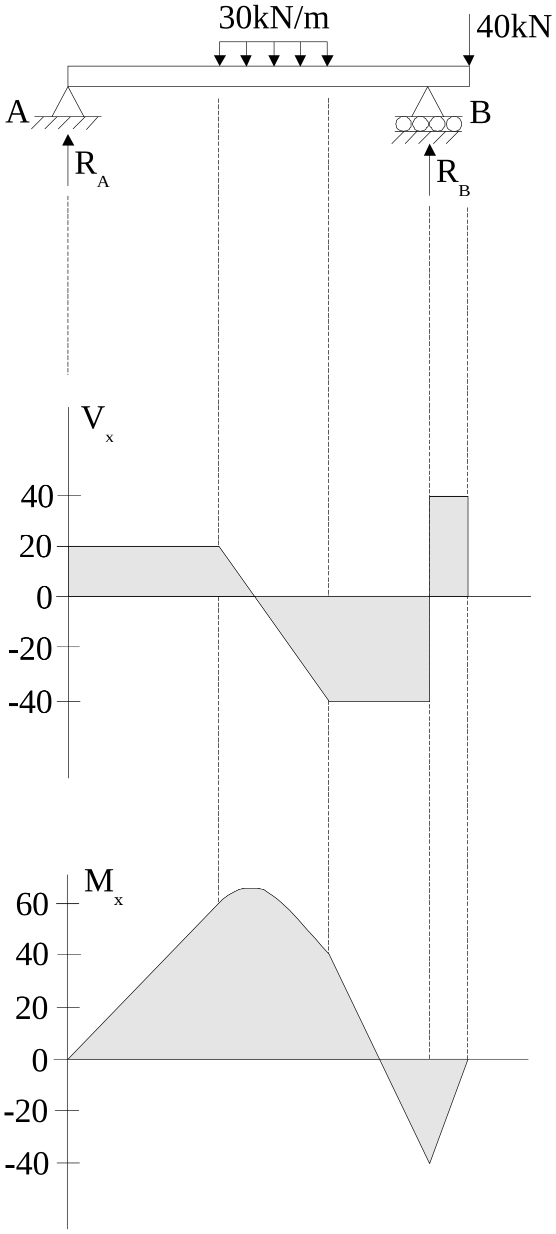


Figure Sol5.1. Shear force and bending moment diagrams for the simply supported beam of Exercise 5.1.

5.2 From the equations of equilibrium the two unknown reactions are

$$R_A = R_B = \frac{WL}{2} = \frac{5\text{x}10^3\text{x}5}{2} = 12.5\text{kN}$$

Taking moments at an arbitrary cut at a distance x from the left hand end

$$M_{xx} + Wx\frac{x}{2} - R_A x = 0 \quad \Rightarrow \quad M_{xx} = R_A x - \frac{Wx^2}{2} = \frac{WLx}{2} - \frac{Wx^2}{2}$$

Differentiating M_{xx} with respect to x and setting dM_{xx}/dx=0 we find that the maximum bending moment occurs at x=L/2. Substituting x=L/2 into M_{xx} then the maximum bending moment is given by

$$M_{\max} = \frac{WL^2}{8} = \frac{5\text{x}10^3\text{x}5^2}{8} = 15{,}625\text{Nm}$$

For the rectangular section the second moment of area is given by, Example 2.3

$$I=\frac{\left(50\text{x}10^{-3}\right)\left(75\text{x}10^{-3}\right)^3}{12}=1.7578\text{x}10^{-6}\,\text{m}^4$$

From (5.30) the maximum tensile and compressive stresses occur at $y=\pm h/2=\pm 37.5$mm

$$s_{\max}=\pm\frac{M_{\max}(h/2)}{I}=\pm\frac{15{,}625\left(37.5\text{x}10^{-3}\right)}{1.7578\text{x}10^{-6}}=\pm 333\text{MPa}$$

5.3 From Example 5.2 the shear force, V_x, at a distance x from the left hand end of the beam is

$$V_x=W\left(\frac{L}{2}-x\right)$$

and attains maximum and minimum values at x=0 and x=L respectively

$$V_{\max}=\frac{WL}{2}=\frac{5\text{x}10^3\text{x}5}{2}=12.5\text{kN}\quad,\quad V_{\min}=-V_{\max}=-12.5\text{kN}$$

From the beam shear formula (5.40) and the rectangular cross-section examined in Example 5.4 then the maximum shear stress at a given section is

$$t_{\max}=\frac{V_x h^2}{8I}$$

and occurs at the neutral axis of the beam. Substituting V_{max} and V_{min} we have

$$t_{\max,\min}=\pm\frac{12.5\text{x}10^3\left(75\text{x}10^{-3}\right)^2}{8\left(1.7578\text{x}10^{-6}\right)}=\pm 5\text{MPa}$$

and are seen to be considerably less than the maximum tensile and compressive bending stresses of Exercise 5.2.

5.4 From the equations of equilibrium the two unknown reactions R_A and R_B are found

$$\sum F^v=0:\quad R_A+R_B=\frac{WL}{2}$$

$$\sum M=0:\quad R_B L-\frac{WL}{2}\left(\frac{2L}{3}\right)=0\ \Rightarrow\ R_B=\frac{WL}{3}\ \Rightarrow\ R_A=\frac{WL}{6}$$

where the centre of gravity of the distributed load acts at x=2L/3. From the equations of equilibrium for the free body diagram of Figure Sol5.4 the bending moment is given by

$$\sum M=0:\quad M_x+\frac{Wx^2}{2L}\left(\frac{x}{3}\right)-R_A x=0\ \Rightarrow\ M_x=\frac{WLx}{6}-\frac{Wx^3}{6L}$$

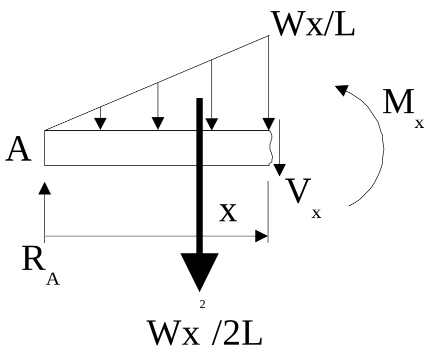


Figure Sol5.4. Free body diagram for the beam of Figure 5.27 cut at a distance x from support A.

Substituting the bending moment into (5.41) we have

$$EIv'' = \frac{WLx}{6} - \frac{Wx^3}{6L}$$

Integrating with respect to x then the slope of the beam is

$$EIv' = \frac{WLx^2}{12} - \frac{Wx^4}{24L} + C_1$$

and integrating once more then the deflection is

$$EIv = \frac{WLx^3}{36} - \frac{Wx^5}{120L} + C_1 x + C_2$$

The boundary condition v=0 at x=0 reveals that C_2=0 and the boundary condition v=0 at x=L gives C_1=-21WL^3/1080. Substituting C_1 and C_2 into the above expressions for v and v' and re-arranging we have

$$v = -\frac{Wx}{360LEI}\left(3x^4 - 10L^2x^2 + 7L^4\right)$$

$$v' = -\frac{W}{360LEI}\left(7L^4 - 30L^2x^2 + 15x^4\right)$$

The maximum deflection, δ_{max}, can be determined from the condition that the slope of the beam will be horizontal at the point of maximum deflection, x_{max}. Substituting v'=0 in the above and re-arranging we have the following quadratic equation for unknown x^2

$$15x^4 - 30L^2x^2 + 7L^4 = 0$$

Solving for x^2

$$x^2 = \left(1 \pm \frac{\sqrt{480}}{30}\right)L^2$$

which gives the two solutions x=1.3154L and x=0.5193L. Since $x \leq L$ then the maximum deflection occurs at x_{max}=0.5193L. Finally, substituting x_{max} into v we arrive at δ_{max}

$$d_{\max} = -v(0.5193L) = 0.00652\frac{WL^4}{EI}$$

5.5 From the equations of equilibrium for the entire beam

$$R_A + R_B - \frac{WL}{2} = 0 \quad , \quad \frac{WL}{2}\frac{3L}{4} - R_A L = 0$$

we find that the reactions R_A and R_B are given by

$$R_A = \frac{3WL}{8} \quad , \quad R_B = \frac{WL}{8}$$

As noted in Example 5.8, when using Macaulay's method, if a distributed load does not extend to the right hand end of the beam then we need to extend and counterbalance the distributed load to the right hand end of the beam. The bending moment at a cut *x*-*x* ($L/2<x<L$) is, Figure 5.27b)

$$M_{xx} + Wx\frac{x}{2} - R_A x - W\{x - L/2\}\frac{\{x - L/2\}}{2} = 0$$

$$\Rightarrow \quad M_{xx} = \frac{3WLx}{8} + \frac{W\{x - L/2\}^2}{2} - \frac{Wx^2}{2}$$

Substituting M_{xx} into the flexure formula (5.41)

$$EIv'' = \frac{3WLx}{8} + \frac{W\{x - L/2\}^2}{2} - \frac{Wx^2}{2}$$

and integrating twice

$$EIv' = \frac{3WLx^2}{16} + \frac{W\{x - L/2\}^3}{6} - \frac{Wx^3}{6} + C_1$$

$$EIv = \frac{3WLx^3}{48} + \frac{W\{x - L/2\}^4}{24} - \frac{Wx^4}{24} + C_1 x + C_2$$

From the two boundary conditions v=0 at x=0 and x=L we find that the constants of integration C_1 and C_2 are

$$C_1 = -\frac{9WL^3}{384} \quad , \quad C_2 = 0$$

and upon substitution into v we arrive at the following expression for the deflection of the entire beam

$$EIv = \frac{WLx^3}{16} + \frac{W\{x - L/2\}^4}{24} - \frac{Wx^4}{24} - \frac{9WL^3 x}{384}$$

Macaulay's method informs us that a term in curly brackets is ignored if it is either zero or negative. Therefore, when $x \le L/2$ we have

$$v = -\frac{Wx}{384EI}\left(16x^3 - 24Lx^2 + 9L^3\right) \quad ; \quad 0 \le x \le \frac{L}{2}$$

5.6 The beam of Figure 5.28 has three unknown reactions (R_A, R_B and M_A) but only two independent equations equilibrium can be written and as a result the beam is statically indeterminate to the first degree. Let reaction R_B be the redundant reaction. From the equations of equilibrium

$$R_A + R_B - WL = 0 \quad \Rightarrow \quad R_A = WL - R_B$$

$$M_A + R_B L - WL\frac{L}{2} = 0 \quad \Rightarrow \quad M_A = \frac{WL^2}{2} - R_B L$$

The bending moment at an arbitrary cut *x*-*x* is

$$M_{xx} + M_A + Wx\frac{x}{2} - R_A x = 0 \quad \Rightarrow \quad M_{xx} = R_A x - M_A - \frac{Wx^2}{2}$$

Substituting R_A and M_A into M_{xx} and using the flexure formula (5.41) we have

$$EIv'' = WLx - R_B x - \frac{WL^2}{2} + R_B L - \frac{Wx^2}{2}$$

Performing two integrations we have

$$EIv' = \frac{WLx^2}{2} - \frac{R_B x^2}{2} - \frac{WL^2 x}{2} + R_B Lx - \frac{Wx^3}{6} + C_1$$

$$EIv = \frac{WLx^3}{6} - \frac{R_B x^3}{6} - \frac{WL^2 x^2}{4} + \frac{R_B Lx^2}{2} - \frac{Wx^4}{24} + C_1 x + C_2$$

Applying the boundary conditions $v'=0$ at $x=0$ and $v=0$ at $x=0$ we find that $C_1=0$ and $C_2=0$. Applying the boundary condition $v=0$ at $x=L$ gives the redundant reaction

$$R_B = \frac{3WL}{8}$$

The unknown reactions R_A and M_A are now given by

$$R_A = \frac{5WL}{8} \quad , \quad M_A = \frac{WL^2}{8}$$

Substituting R_B, C_1 and C_2 into v and collecting terms then the deflection for the beam is given by

$$v = -\frac{Wx^2}{48EI}\left(3L^2 + 2x^2 - 5Lx\right)$$

5.7 The beam has three unknown reactions (R_A, R_B and M_A) but only two independent equations of equilibrium and is therefore statically indeterminate to the first degree. Letting R_B be the redundant reaction then from the equations of equilibrium

$$R_A = WL - R_B \quad , \quad M_A = \frac{WL^2}{2} - R_B L$$

Using the method of superposition we now proceed to apply the distributed load W to the released beam, Figure Sol5.7b), and the redundant reaction R_A to the released beam, Figure Sol5.7c). Since the deflection of the original beam is zero at end B then we have the following compatibility equation

$$d_B = d_{B1} - d_{B2} = 0$$

The force-displacement relations give the deflections δ_{B1} and δ_{B2}

$$d_{B1} = \frac{WL^4}{8EI} \quad , \quad d_{B2} = \frac{R_B L^3}{3EI}$$

which upon substituting into the compatibility equation gives

$$d_B = \frac{WL^4}{8EI} - \frac{R_B L^3}{3EI} = 0$$

and solving for R_B

$$R_B = \frac{3WL}{8}$$

The remaining reactions R_A and M_A are found from the above equilibrium equations

$$R_A = \frac{5WL}{8} \quad , \quad M_A = \frac{WL^2}{8}$$

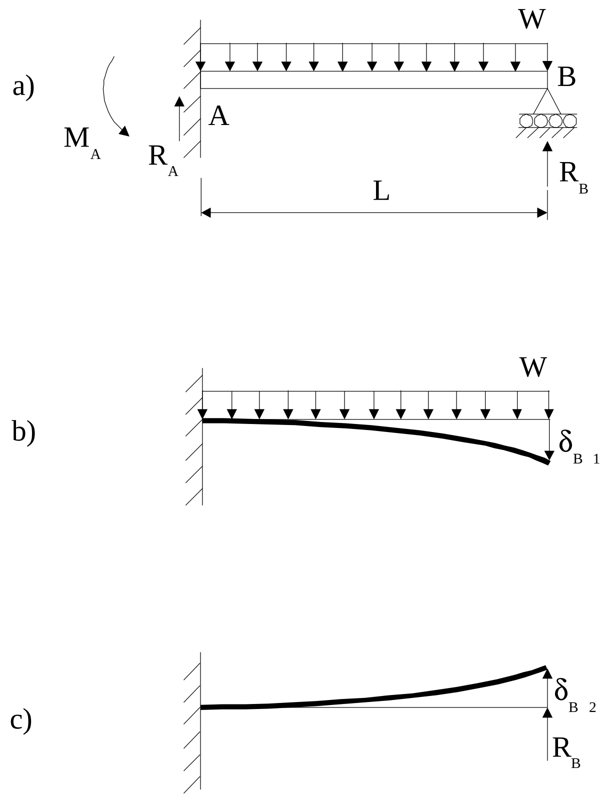


Figure Sol5.7. Propped cantilever beam of Exercise 5.7. a) The propped cantilever beam subject to a uniformly distributed load. b) The released beam with redundant reaction R_B and load W applied. c) The released beam with R_B applied.

Chapter 6 Solutions

6.1 With $I=\pi d^4/64$ for a solid circular cross-section of diameter d and substituting I into (6.10) and re-arranging for d we have

$$d=\left(\frac{64P_{cr}L^2}{\pi^3E}\right)^{1/4}=\left(\frac{64\text{x}200\text{x}10^3\text{x}4^2}{\pi^3\text{x}210\text{x}10^9}\right)^{1/4}=75\text{mm}$$

6.2 The second moments of area with respect to centroidal coordinates (x,y) for the square, circle and equilateral triangle shown in Figure Sol6.2 are

square: $I=I_{xx}=I_{yy}=\dfrac{Ab^2}{12}$, $A=b^2$, $r^2=\dfrac{I}{A}=\dfrac{A}{12}\approx 0.08\dot{3}A$

circle: $I=I_{xx}=I_{yy}=\dfrac{Ab^2}{4}$, $A=\pi b^2$, $r^2=\dfrac{I}{A}=\dfrac{A}{4\pi}\approx 0.0797A$

equilateral triangle: $I=I_{xx}=I_{yy}=\dfrac{Ab^2}{36}$, $A=\dfrac{\sqrt{3}}{4}b^2$, $r^2=\dfrac{I}{A}=\dfrac{A}{8\sqrt{3}}\approx 0.0722A$

From (6.14) σ_{cr} is given by

$$\sigma_{cr}=\frac{\pi^2Er^2}{L^2}$$

with σ_{cr} seen to be proportional to r^2. Therefore, the struts from largest to smallest σ_{cr} are the square, circular and equilateral triangle cross-sections.

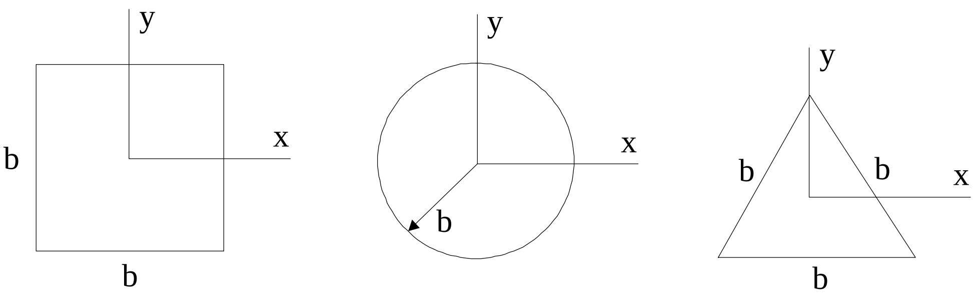


Figure Sol6.2. Square, circle and equilateral triangle.

6.3 With reference to Figure Sol6.3 the second moments of area I_x and I_y are, (2.5) and (2.6)

$$I_x=\iint_A y^2dA=4\int(a\cos\theta)^2(tds)=4\int_0^{\pi/2}(a\cos\theta)^2(tad\theta)=\pi a^3t$$

$$I_y=\iint_A x^2dA=4\int(a\sin\theta)^2(tds)=4\int_0^{\pi/2}(a\sin\theta)^2(tad\theta)=\pi a^3t$$

with I_x=I_y due to symmetry. The area of the section is

$$A=\iint_A dA=4\int tds=4\int_0^{\pi/2}t(ad\theta)=2\pi at$$

Finally, the radius of gyration is, (6.12)

$$r=\sqrt{\frac{I}{A}}=\sqrt{\frac{\pi a^3t}{2\pi at}}=\frac{a}{\sqrt{2}}$$

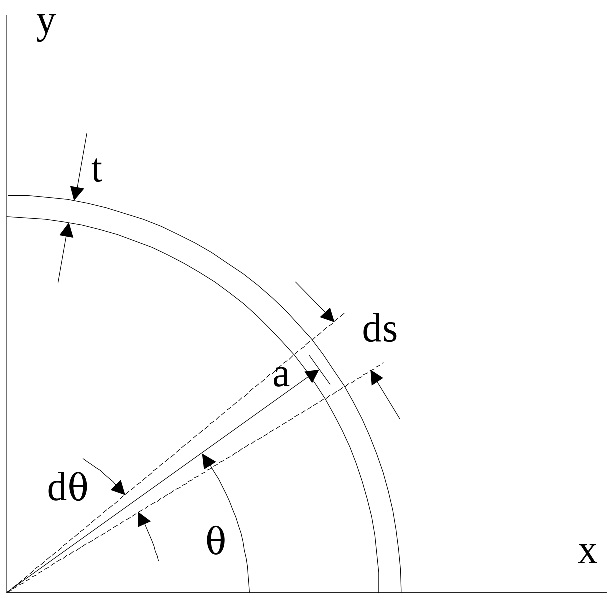


Figure Sol6.3. One quadrant of a circular tubular cross-section.

6.4 From the equations of equilibrium the compressive load acting on the strut is equal to $\sqrt{3}W$. The cross-sectional area and second moment of area of the strut are

$$A = \left(50\text{x}10^{-3}\right)^2 = 2.5\text{x}10^{-3}\,\text{m}^2$$

$$I = \frac{1}{12}\left(50\text{x}10^{-3}\right)\left(50\text{x}10^{-3}\right)^3 = 5.208\dot{3}\text{x}10^{-7}\,\text{m}^4$$

From (6.12) the radius of gyration is

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{5.2083\text{x}10^{-7}}{2.5\text{x}10^{-3}}} = 0.0144\text{m}$$

The strut is built-in at one end and pin-jointed at the other end where the load W acts so that from (6.42) the effective length of the strut is L_e=0.7L=1.75m and the critical buckling load is, (6.31)

$$P_{cr} = \frac{\pi^2 EA}{\left(L_e / r\right)^2} = \frac{\pi^2\text{x}210\text{x}10^9\text{ x}2.5\text{x}10^{-3}}{\left(1.75/0.0144\right)^2} = 351\text{kN}$$

Thus, the strut will fail due to buckling when W exceeds 351kN.

6.5 The area and second moment of area of the rectangular tube are

$$A = \left(0.2\text{x}0.1\right) - \left(0.18\text{x}0.08\right) = 5.6\text{x}10^{-3}\,\text{m}^2$$

$$I = \frac{1}{12}\left[0.2\text{x}0.1^3 - 0.18\text{x}0.08^3\right] = 8.98\dot{6}\text{x}10^{-6}\,\text{m}^4$$

From (6.32) the critical buckling load is

$$P_{cr} = \frac{4\pi^2 EI}{L^2} = \frac{4\pi^2\text{x}210\text{x}10^9\text{ x}8.986\text{x}10^{-6}}{5^2} = 993.3\text{kN}$$

and the critical stress is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{993.3\text{x}10^3}{5.6\text{x}10^{-3}} = 177.4\text{MPa}$$

6.6 The stress and eccentricity ratio are

$$\frac{P}{A} = \frac{350\text{x}10^3}{3270\text{x}10^{-6}} = 107\text{MPa}$$

$$\frac{ey}{r^2} = \frac{eA}{S} = \frac{0.025\text{x}3270\text{x}10^{-6}}{144\text{x}10^{-6}} = 0.5677$$

The critical buckling load is, (6.10)

$$P_{cr} = \frac{p^2 EI}{L^2} = \frac{p^2\text{x}210\text{x}10^9\text{ x}11\text{x}10^{-6}}{5^2} = 912\text{kN}$$

From (6.52) and (6.57) the maximum deflection and stress are

$$d = e\left[\sec\left(\frac{p}{2}\sqrt{\frac{P}{P_{cr}}}\right) - 1\right] = 19.5\text{mm}$$

$$s_{\max} = \frac{P}{A}\left[1 + \frac{ey}{r^2}\sec\left(\frac{p}{2}\sqrt{\frac{P}{P_{cr}}}\right)\right] = 215\text{MPa}$$

6.7 From Table 6.2 the constant *a* is equal to 1/7500 and the slenderness ratio is

$$l = \frac{L}{r} = \frac{2}{39.6\text{x}10^{-3}} = 50.505$$

Therefore, from (6.61) the critical stress according to the Rankine-Gordon formula is

$$s_R = \frac{s_Y}{1 + al^2} = \frac{300\text{x}10^6}{1 + \frac{50.505^2}{7500}} = 224\text{MPa}$$

Chapter 7 Solutions

7.1 The σ_{xx} stress is

$$\sigma_{xx} = \frac{P}{A} = -\frac{75\text{x}10^{3}}{1500\text{x}10^{-6}} = -50\text{MPa}$$

From (7.8) and (7.12) the local direct and shear stresses on the cut plane *ab* are

$$\sigma_{x'x'} = \sigma_{xx}\frac{1}{2}\left(1+\cos 2\theta\right) = \sigma_{xx}\cos^{2}\theta = -37.5\text{MPa}$$

$$\sigma_{y'y'} = \sigma_{xx}\frac{1}{2}\left(1-\cos 2\theta\right) = -12.5\text{MPa}$$

$$\tau_{x'y'} = -\sigma_{xx}\frac{1}{2}\sin 2\theta = -\sigma_{xx}\sin\theta\cos\theta = 21.65\text{MPa}$$

7.2 From the stress transformation equations we have

$$\sigma_{x'x'} = 195\text{MPa} \quad , \quad \sigma_{y'y'} = 105\text{MPa} \quad , \quad \tau_{x'y'} = 50\text{MPa}$$

observing that the sum of the global and local stresses are equal, (7.13)

$$\sigma_{xx} + \sigma_{yy} = \sigma_{x'x'} + \sigma_{y'y'}$$

7.3 The centre *C*, point *A*, point *B* and radius *R* of Mohr's circle are, see §7.5.1

$$C = \left[\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right] = \left[-25,0\right]$$

$$A\left(\theta = 0^{\circ}\right) = \left(\sigma_{xx}, \tau_{xy}\right) = \left(50,-50\right)$$

$$B\left(\theta = 90^{\circ}\right) = \left(\sigma_{yy}, -\tau_{xy}\right) = \left(-100,50\right)$$

$$R = \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^{2} + \tau_{xy}^{2}} = 90.14$$

Mohr's circle can now be plotted and is shown in Figure Sol7.3.

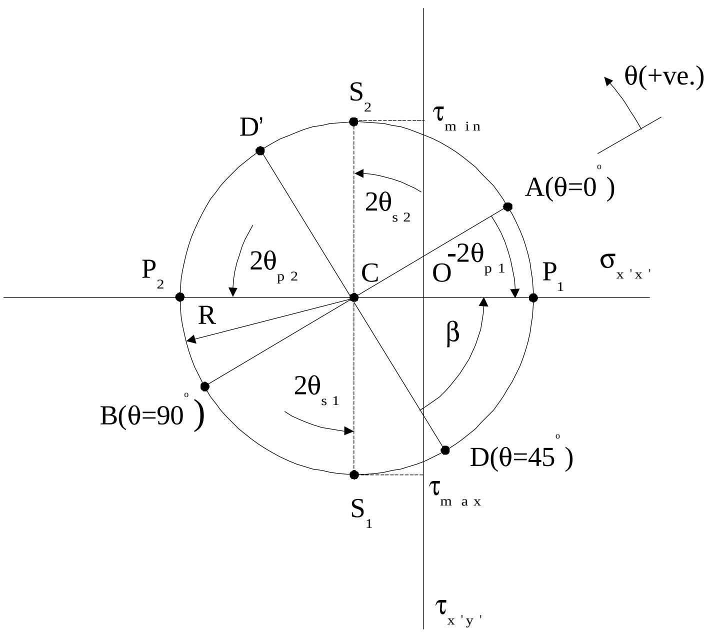


Figure Sol7.3. Mohr's circle for the stress element of Exercise 7.3.

The principal stresses and associated planes are

$$\sigma_1 = OC + R = -25 + 90.14 = 65.14$$

$$\sigma_2 = OC - R = -25 - 90.14 = -115.14$$

$$2\theta_{p2} = 180^\circ - \angle ACP_1 = 180^\circ - \tan^{-1}\left(\frac{2}{3}\right) = 146.31^\circ \quad ; \quad \theta_{p2} = 73.16^\circ$$

$$2\theta_{p1} = 180^\circ + 2\theta_{p2} \quad ; \quad \theta_{p1} = 163.16^\circ \quad \text{or} \quad -16.84^\circ$$

The maximum and minimum shear stresses and associated planes are

$$\tau_{\max} = R = 90.14$$

$$\tau_{\min} = -R = -90.14$$

$$2\theta_{s2} = \angle ACS_2 = \tan^{-1}\left(\frac{3}{2}\right) = 56.31^\circ \quad ; \quad \theta_{s2} = 28.15^\circ$$

$$2\theta_{s1} = 180^\circ + 2\theta_{s2} \quad ; \quad \theta_{s1} = 118.15^\circ$$

The stresses on the plane θ=-45° are represented by points *D* and *D′* on Mohr's circle shown in Figure Sol7.3. With angle β given by

$$\beta = 2\theta - \angle ACP_1 = 90^\circ - \tan^{-1}\left(\frac{2}{3}\right) = 90^\circ - 33.69^\circ = 56.31^\circ$$

$$\sigma_{x'x'}(D) = OC + R\cos\beta = -25 + 90.14\cos 56.31^\circ = 25$$

$$\tau_{x'y'}(D) = R\sin\beta = 90.14\sin 56.31^\circ = 75$$

$$\sigma_{y'y'} = \sigma_{x'x'}(D') = OC - R\cos\beta = -25 - 90.14\cos 56.31^\circ = -75$$

7.4 Mohr's circle can be schematically constructed by positioning the centre, *C*, of the circle and determining its radius, *R*, as follows for the three requested cases:

a) Uniaxial ($\sigma_{xx}=\sigma$, $\sigma_{yy}=\tau_{xy}=0$)

$$C = \left[\frac{\sigma_{xx} + \sigma_{yy}}{0}, 0\right] = \left[\frac{\sigma}{2}, 0\right]$$

$$R = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = \frac{\sigma}{2}$$

b) Equi-biaxial ($\sigma_{xx}=\sigma_{yy}=\sigma$, $\tau_{xy}=0$)

$$C = [\sigma, 0]$$

$$R = 0$$

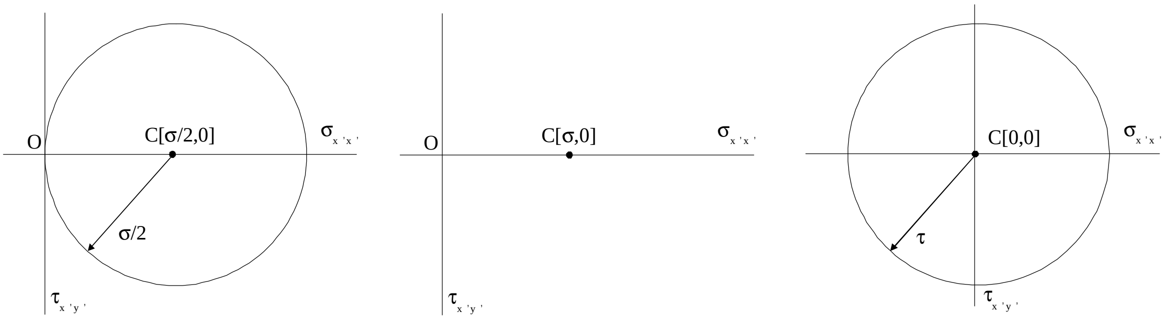
noting that Mohr's circle reduces to a point.

c) Pure Shear ($\sigma_{xx}=\sigma_{yy}=0$, $\tau_{xy}=\tau$)

$$C = [0,0]$$

$$R = \tau$$

Each of the above cases is correspondingly illustrated in Figure Sol7.4.



a) b) c)

Figure Sol7.4. Mohr’s circle for the cases of a) uniaxial, b) equi-biaxial and c) pure shear loadings.

7.5 (i) With I_1, I_2 and θ_p given by

$$I_{1,2} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \quad , \quad \tan 2J_p = -\frac{2I_{xy}}{I_x - I_y}$$

and since $I_x = I_y$ then

$$I_1 = I_x + I_{xy} = \left(\frac{p}{16} - \frac{4}{9p} + \frac{1}{8} - \frac{4}{9p}\right) r^4 = \left(\frac{9p^2 + 18p - 128}{144p}\right) r^4$$

$$I_2 = I_x - I_{xy} = \left(\frac{p}{16} - \frac{4}{9p} + \frac{1}{8} + \frac{4}{9p}\right) r^4 = \left(\frac{p - 2}{16}\right) r^4$$

$$\tan 2J_p = \infty \quad ; \quad 2J_p = 90^\circ \quad ; \quad J_p = 45^\circ$$

(ii) With r=10mm then from the I_x, I_y and I_{xy} expressions given

$$I_x = I_y = 549\text{mm}^4 \quad , \quad I_{xy} = -165\text{mm}^4$$

and from the transformation equations

$$I_{x',y'} = \left(\frac{I_x + I_y}{2}\right) \pm \left(\frac{I_x - I}{2}\right)\cos 2J \mp I_{xy}\sin 2J = I_x \mp I_{xy}\sin 2J = 692\text{mm}^4 \text{ and } 406\text{mm}^4$$

$$I_{x'y'} = I_{xy}\cos 2J = -82.5\text{mm}^4$$

7.6 With θ=30° then cos2θ=1/2 and sin2θ=√3/2 and substituting I_x, I_y and I_{xy} into (7.39) and (7.41) we find that $I_{x'}=I_y$, $I_{y'}=I_x$ and $I_{x'y'}=I_{xy}$ as required.

7.7 From (7.54) the shear modulus, G, for the aluminium alloy and steel are

$$G_{al} = \frac{E_{al}}{2(1+n_{al})} = \frac{70}{2(1+0.33)} = 26\text{GPa}$$

$$G_{steel} = \frac{E_{steel}}{2(1+n_{steel})} = \frac{210}{2(1+0.3)} = 81\text{GPa}$$

Chapter 8 Solutions

8.1 Since θ is taken as positive for an anticlockwise rotation then θ=-30° in the present case. From (8.10) the local strains are

$$\varepsilon_{x'x',y'y'} = \left(\frac{500+(-300)}{2}\right)\text{x}10^{-6} \pm \left(\frac{500-(-300)}{2}\right)\text{x}10^{-6}\cos(-60^\circ) \pm \frac{200\text{x}10^{-6}}{2}\sin(-60^\circ)$$

$$= 213\text{x}10^{-6}, -13.4\text{x}10^{-6}$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{500-(-300)}{2}\right)\text{x}10^{-6}\sin(-60^\circ) + \frac{200\text{x}10^{-6}}{2}\cos(-60^\circ) = 396.4\text{x}10^{-6}$$

so that $\gamma_{x'y'}$ is equal to 793x10^{-6}.

8.2 From (8.11) the principal strains ε_1 and ε_2 are

$$\varepsilon_{1,2} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = -190\text{x}10^{-6}, -360\text{x}10^{-6}$$

From (8.11) we obtain the planes of the principal strains

$$2\theta_p = \tan^{-1}\left(\frac{80}{-350-(-200)}\right) = \tan^{-1}\left(\frac{8}{15}\right)$$

which has the two roots of θ_p=-14° and 76° for θ_p in the range 0≤θ_p≤180°. To establish which angle is associated with either ε_1 and ε_2 then let us examine, say, θ_p=-14° for $\varepsilon_{x'x'}$, (8.10)

$$\varepsilon_{x'x'}(-14^\circ) = \frac{-350+(-200)}{2}\text{x}10^{-6} + \frac{-350-(-200)}{2}\text{x}10^{-6}\cos(-28^\circ) + \frac{80}{2}\text{x}10^{-6}\sin(-28^\circ)$$

$$= -360\text{x}10^{-6}$$

observing that $\varepsilon_{x'x'}$ (-14°)=ε_2 so we conclude that θ_{p2}=-14° and θ_{p1}=76°. The principal strains are illustrated graphically in Figure Sol8.2a).

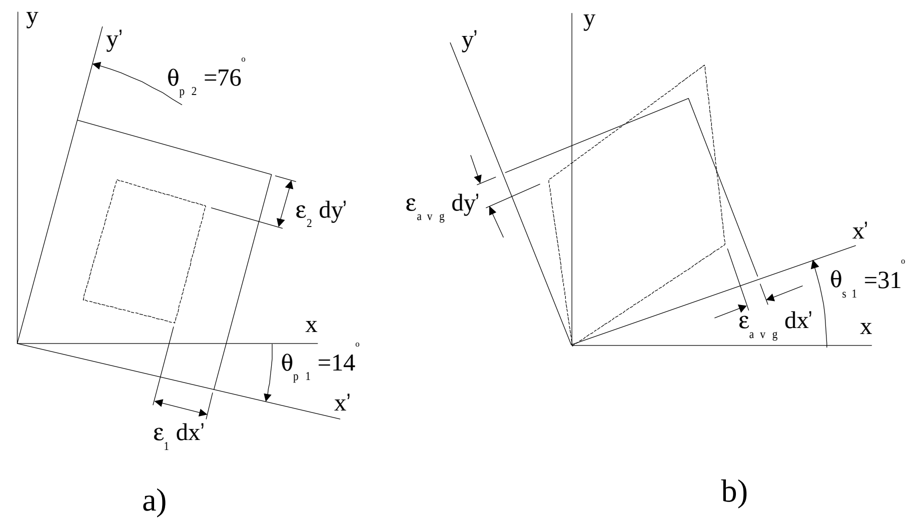


Figure Sol8.2. Schematic illustration of strains. a) Principal stress element. b) Maximum shear strain element.

The maximum shear strain is, (8.12)

$$\left(\frac{\gamma_{x'y'}}{2}\right)_{\max} = \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2} = \sqrt{\left(\frac{-350-(-200)}{2}\right)^2+\left(\frac{80}{2}\right)^2} = 85\text{x}10^{-6}$$

and therefore γ_{max}=170x10^{-6} and is illustrated graphicaly in Figure Sol8.2b).

8.3 With θ_a=0°, θ_b=60° and θ_c=120° we find that the system of equations (8.18) reduces to

$$60\text{x}10^{-6} = \varepsilon_{xx}$$
$$135\text{x}10^{-6} = 0.25\varepsilon_{xx} + 0.75\varepsilon_{yy} + 0.433\gamma_{xy}$$
$$264\text{x}10^{-6} = 0.25\varepsilon_{xx} + 0.75\varepsilon_{yy} - 0.433\gamma_{xy}$$

which result in the global strains ε_{xx}=60x10^{-6}, ε_{yy}=246x10^{-6} and γ_{xy}=-149x10^{-6}.

We will use the transformations equations to determine the principal strains and their associated planes. From (8.11) the principal strains are

$$\varepsilon_{1,2} = \left(\frac{60+246}{2}\right) \pm \sqrt{\left(\frac{6-246}{2}\right)^2+\left(\frac{-149}{2}\right)^2} = 272\text{x}10^{-6}\ ,\ 34\text{x}10^{-6}$$

and with principal planes

$$\theta_p = \frac{1}{2}\tan^{-1}\left(\frac{\gamma_{xy}}{\varepsilon_{xx}-\varepsilon_{yy}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{-149}{60-246}\right) = \frac{1}{2}\tan^{-1}(0.8) = 19.33^\circ$$

Inserting 19.33° into (8.10) we find that θ_{p2}=19.33° and therefore θ_{p1}=90°+θ_{p2}=109.33°. These results can be compared with Example 8.2 which alternatively determined the principal strains and planes using Mohr's circle.

8.4 Points *A* and *B*, centre *C* and radius *R* of Mohr's circle for the in-plane strains ε_{xx}=250x10^{-6}, ε_{yy}=-150x10^{-6} and γ_{xy}=120x10^{-6} are as follows

$$A(\theta = 0^\circ) = (\varepsilon_{xx}, \gamma_{xy}/2) = (250\text{x}10^{-6}, 60\text{x}10^{-6})$$
$$B(\theta = 90^\circ) = (\varepsilon_{yy}, -\gamma_{xy}/2) = (-150\text{x}10^{-6}, -60\text{x}10^{-6})$$
$$C = [\varepsilon_{avg}, 0] = [50\text{x}10^{-6}, 0]$$
$$R = \sqrt{\left(\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\right)^2+\left(\frac{\gamma_{xy}}{2}\right)^2} = 209\text{x}10^{-6}$$

with Mohr's strain circle illustrated in Figure Sol8.4. From Mohr's circle we find the principal strains and planes

$$\varepsilon_1 = OC + R = (50+209)x10^{-6} = 259x10^{-6}$$
$$\varepsilon_2 = OC - R = (50-209)x10^{-6} = -159x10^{-6}$$
$$2\theta_{p1} = \tan^{-1}\left(\frac{60}{250-50}\right) = 16.7^\circ \quad ; \quad \theta_{p1} = 8.35^\circ$$
$$2\theta_{p2} = 180^\circ + 2\theta_{p1} \quad ; \quad \theta_{p2} = 98.35^\circ$$

The maximum shear strain $(\gamma/2)_{max}$=*R* so that γ_{max}=418x10^{-6}. From Mohr's circle the planes of the maximum and minimum shear strains are

$$2\theta_{s1} = -(90^\circ - 2\theta_{p1}) = -73.3^\circ \quad ; \quad \theta_{s1} = -36.65^\circ$$
$$2\theta_{s2} = 180^\circ - \|2\theta_{s1}\| = 106.7^\circ \quad ; \quad \theta_{s2} = 53.35^\circ$$

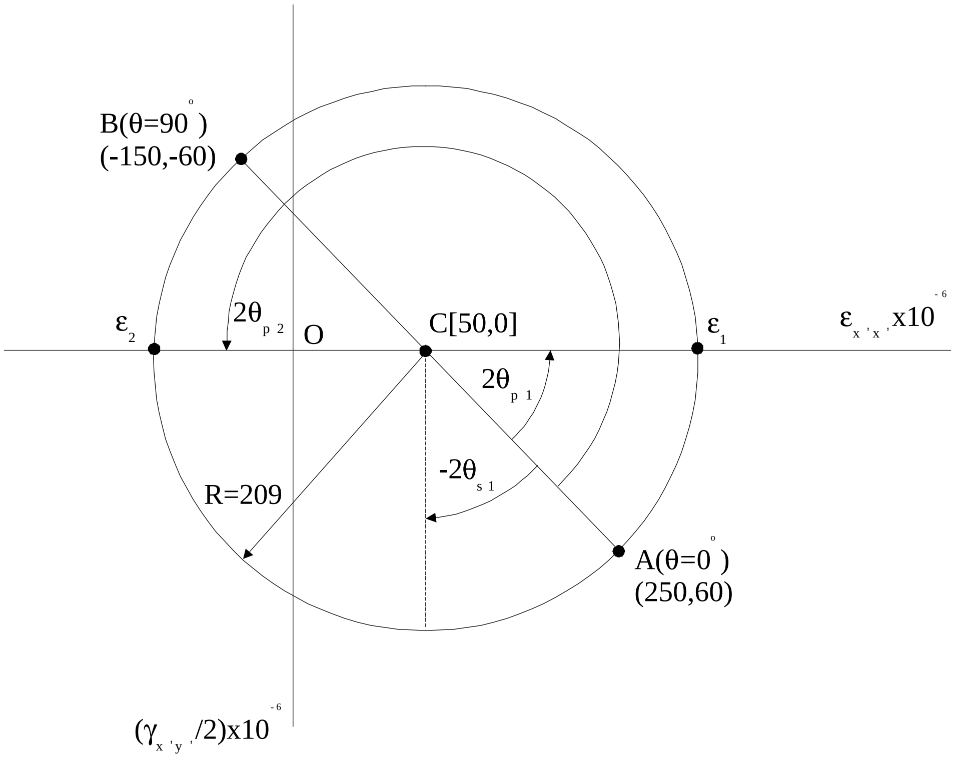


Figure Sol8.4. Mohr's circle for the in-plane strains ε_{xx}=250x10^{-6}, ε_{yy}=-150x10^{-6} and γ_{xy}=120x10^{-6}.

8.5 The average normal strain ε_{avg}, centre *C*, points *A* and *B* and radius *R* of Mohr's strain circle are

$$\varepsilon_{avg} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} = -200\text{x}10^{-6}$$

$$C = [\varepsilon_{avg}, 0] = [-200x10^{-6}, 0]$$

$$A(\theta = 0^\circ) = (\varepsilon_{xx}, \gamma_{xy}/2) = (-300\text{x}10^{-6}, 50\text{x}10^{-6})$$

$$B(\theta = 90^\circ) = (\varepsilon_{yy}, -\gamma_{xy}/2) = (-100\text{x}10^{-6}, -50\text{x}10^{-6})$$

$$R = \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 111.8\text{x}10^{-6}$$

Mohr's circle can now be constructed and is shown in Figure Sol8.5. Since we are required to determine the strain components on an element that is rotated by 20° in a clockwise direction then θ=-20°. From Figure Sol8.5 the principal plane θ_{p2} is given by

$$\theta_{p2} = -\frac{1}{2}\tan^{-1}\left(\frac{50}{300-200}\right) = -\frac{1}{2}\tan^{-1}\frac{1}{2} = -13.28^\circ$$

so that β=2θ-2θ_{p2}=40°-26.57°=13.43°. From triangle *DEC* we find the local strains

$$\varepsilon_{x'x'} = -(OC + R\cos\beta) = -(200 + 111.8\cos 13.43^\circ)\text{x}10^{-6} = -309\text{x}10^{-6}$$

$$\frac{\gamma_{x'y'}}{2} = -R\sin\beta = -111.8\text{x}10^{-6}\sin 13.43^\circ = -25.97\text{x}10^{-6} \quad ; \quad \gamma_{x'y'} = 51.93\text{x}10^{-6}$$

and from triangle *D′FC* we find the remaining local strain $\varepsilon_{y'y'}$

$$\varepsilon_{y'y'} = OC + R\cos\beta = (-200 + 111.8\cos 13.43^\circ)x10^{-6} = -91.26\text{x}10^{-6}$$

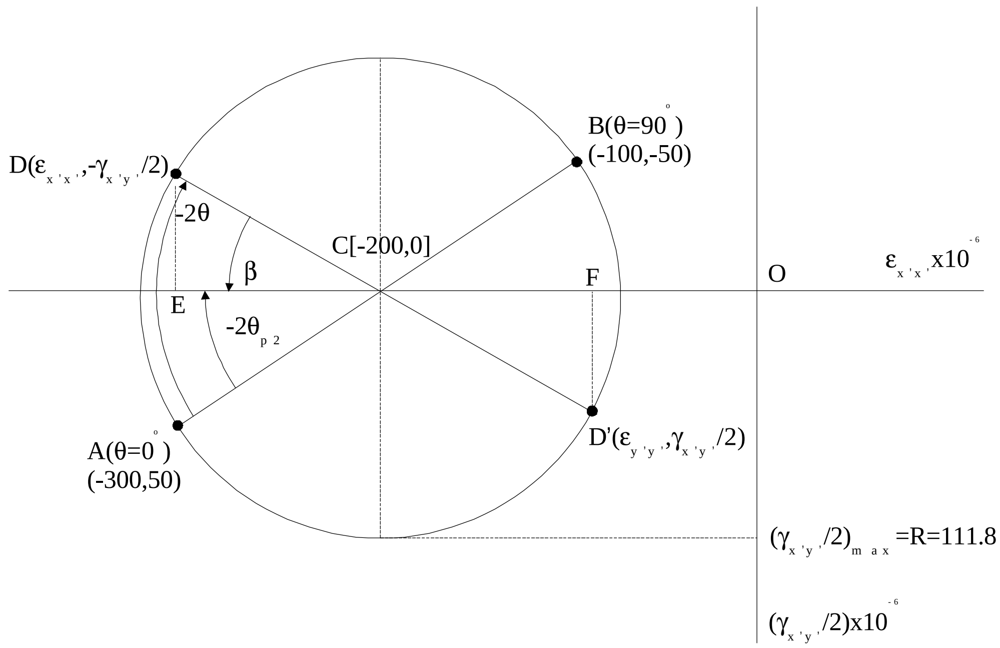


Figure Sol8.5. Mohr's circle for the in-plane strains ε_{xx}=300x10^{-6}, ε_{yy}=-100x10^{-6} and γ_{xy}=100x10^{-6}.

8.6 From (8.10) the local strains are

$$\varepsilon_{x'x',y'y'} = \left(\frac{200+400}{2}\right)\text{x}10^{-6} \pm \left(\frac{200-400}{2}\right)\text{x}10^{-6}\cos 80^\circ \pm \frac{100\text{x}10^{-6}}{2}\sin 80^\circ$$

$$= 332\text{x}10^{-6}\ ,\ 268\text{x}10^{-6}$$

$$\frac{\gamma_{x'y'}}{2} = -\left(\frac{200-400}{2}\right)\text{x}10^{-6}\sin 80^\circ + \frac{100\text{x}10^{-6}}{2}\cos 80^\circ = 107\text{x}10^{-6}$$

so that $\gamma_{x'y'}$ is equal to 214x10^{-6}.

From the Hookian equations (1.16) we have

$$\varepsilon_{x'x'} = \frac{1}{E}\left[\sigma_{x'x'} - \nu\sigma_{y'y'}\right]\quad , \quad \varepsilon_{y'y'} = \frac{1}{E}\left[\sigma_{y'y'} - \nu\sigma_{x'x'}\right]\quad , \quad \gamma_{x'y'} = \frac{\tau_{x'y'}}{G}$$

Solving these for the in-plane stresses then we have

$$\sigma_{x'x'} = \frac{E}{1-\nu^2}\left[\varepsilon_{x'x'} + \nu\varepsilon_{y'y'}\right] = \frac{210\text{x}10^9}{1-0.3^2}\left[332 + 0.3\text{x}268\right]\text{x}10^{-6} = 95\text{MPa}$$

$$\sigma_{y'y'} = \frac{E}{1-\nu^2}\left[\varepsilon_{y'y'} + \nu\varepsilon_{x'x'}\right] = \frac{210\text{x}10^9}{1-0.3^2}\left[268 + 0.3\text{x}332\right]\text{x}10^{-6} = 85\text{MPa}$$

$$\tau_{x'y'} = G\gamma_{x'y'} = \frac{E}{2(1+\nu)}\gamma_{x'y'} = \frac{210\text{x}10^9}{1(1+0.3)}214\text{x}10^{-6} = 17\text{MPa}$$

8.7 With θ_a=0°, θ_b=45° and θ_c=90° then the system of equations (8.18) reduce to

$$65\text{x}10^{-6} = \varepsilon_{xx}$$

$$95\text{x}10^{-6} = 0.5\varepsilon_{xx} + 0.5\varepsilon_{yy} + 0.5\gamma_{xy}$$

$$25\text{x}10^{-6} = \varepsilon_{yy}$$

which result in the global strains ε_{xx}=65x10^{-6}, ε_{yy}=25x10^{-6} and γ_{xy}=100x10^{-6}. From (8.11) the principal strains are

$$\varepsilon_{1,2} = \frac{\varepsilon_{xx} + \varepsilon_{yy}}{2} \pm \sqrt{\left(\frac{\varepsilon_{xx} - \varepsilon_{yy}}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2} = 98.85\text{x}10^{-6} \;\; , \;\; -8.85\text{x}10^{-6}$$

and from (8.11) we obtain the planes of the principal strains

$$2\theta_p = \tan^{-1}\left(\frac{100}{65-25}\right) = \tan^{-1}(2.5) = 68.2^\circ$$

which has the two roots of θ_p=34.1° and 124.1° for θ_p in the range 0≤θ_p≤180°.

Chapter 9 Solutions

9.1 From (9.10) the in-plane strains are

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] \quad , \quad \varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right]$$

and substituting into the strain energy density, (9.9), we have

$$U_0 = \frac{1}{2}\left[\sigma_{xx}\varepsilon_{xx} + \sigma_{yy}\varepsilon_{yy}\right] = \frac{1}{2E}\left[\sigma_{xx}^2 - 2\nu\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2\right]$$

Integrating U_0 throughout the entire volume of the plate then the strain energy is given by

$$U = \int_V U_0 dV = \frac{V}{2E}\left(\sigma_{xx}^2 - 2\nu\sigma_{xx}\sigma_{yy} + \sigma_{yy}^2\right)$$

9.2 From (9.4) the strain energy density, U_0, is equal to $\sigma\varepsilon/2$ so that the strain energy is given by

$$U = \int_V U_0 dV = \int_0^L \frac{\sigma^2(x)}{2E} A(x)dx$$

where A is the cross-sectional area of the bar and is a function of x. With $\sigma(x)=W/A(x)$ then U is

$$U = \frac{W^2}{2E}\int_0^L \frac{dx}{A(x)}$$

It remains to find $A(x)$. Linearly interpolating across the length of the bar from d_1 to d_2 then

$$A(x) = \pi y^2 = \pi\left[\frac{d_1}{2} + \left(\frac{d_2}{2} - \frac{d_1}{2}\right)\frac{x}{L}\right]^2 = \frac{\pi d_1^2}{4}\left[1 + \left(\frac{d_2}{d_1} - 1\right)\frac{x}{L}\right]^2$$

and substituting $A(x)$ into U

$$U = \frac{2W^2}{E\pi d_1^2}\int_0^L \frac{dx}{\left[1 + \left(\frac{d_2}{d_1} - 1\right)\frac{x}{L}\right]^2}$$

The integral is seen to be of the following general form

$$\int \frac{dx}{(ax+b)^2} = \int (ax+b)^{-2} dx = -\frac{1}{a}(ax+b)^{-1}$$

Performing the integration then U is found to be

$$U = \frac{2W^2}{E\pi d_1^2}\left\{-\frac{L}{d_2/d_1 - 1}\left[\frac{1}{1 + (d_2/d_1 - 1)(x/L)}\right]_0^L\right\} = \frac{2W^2 L}{E\pi d_1 d_2}$$

as required.

From Castigiliano's second theorem then the displacement of the bar is, (9.59)

$$\delta = \frac{\partial U}{\partial W} = \frac{4WL}{E\pi d_1 d_2}$$

When $d_1=d_2=d$ then the bar is of constant cross-section and δ is given by

$$\delta = \frac{4WL}{E\pi d^2}$$

and is seen to agree with $\delta=WL/AE$ where $A=\pi d^2/4$.

9.3 Resolving forces vertically at joint *B* then the force, *F*, in each member is

$$F = \frac{P}{2\cos\theta}$$

and the length, *L*, of each member is *d*/cosθ. Considering member *BC* then the strain energy is, (9.12)

$$U_{BC} = \int_V U_o dV = \int_0^L \left(\frac{\sigma_{xx}^2}{2E}\right) A dx = \frac{\sigma_{xx}^2 AL}{2E}$$

with *x* taken along the member axis. Substituting for σ_{xx} and *L* then

$$U_{BC} = \frac{P^2 d}{8EA\cos^3\theta}$$

Due to symmetry $U_{AB}=U_{BC}$ so that the total strain energy, *U*, of the frame is

$$U = U_{AB} + U_{BC} = 2U_{BC} = \frac{P^2 d}{4EA\cos^3\theta}$$

From Castigilano's second theorem (9.59) the displacement, δ_B, at joint *B* is

$$\delta_B = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P}\left(\frac{P^2 d}{4EA\cos^3\theta}\right) = \frac{Pd}{2EA\cos^3\theta}$$

9.4 Re-arranging (9.49) for applied load *W*

$$W = \frac{\delta d^4 \cos\alpha}{8D^3 n\left[\frac{\cos^2\alpha}{G} + \frac{2\sin^2\alpha}{E}\right]} = \frac{10x5^4\cos 25^\circ}{8\text{x}50^3\text{x}10\left[\frac{\cos^2 25^\circ}{85\text{x}10^3} + \frac{2\sin^2 25^\circ}{226.1\text{x}10^3}\right]} = 50.4\text{N}$$

The shear stress is given by (9.51)

$$\tau = \frac{M_x(d/2)}{\pi d^4/32} = \frac{(WD/2)\cos\alpha(d/2)}{\pi d^4/32} = \frac{50.4(50/2)\cos 25^\circ(5/2)}{\pi 5^4/32} = 46.53\text{MPa}$$

and the bending stress is given by (9.52)

$$\sigma = \frac{M_y(d/2)}{\pi d^4/64} = \frac{(WD/2)\sin\alpha(d/2)}{\pi d^4/64} = \frac{50.4(50/2)\sin 25^\circ(5/2)}{\pi 5^4/64} = 43.39\text{MPa}$$

9.5 Letting l_s(=*nd*) denote the solid length then from (9.40) we have

$$k = \frac{W}{\delta} = \frac{Gd^4}{8D^3 n} \quad \Rightarrow \quad 2.5 = \frac{45\text{x}10^9 d^4}{8D^3(l_s/d)} \quad \Rightarrow \quad D^3 = 45\text{x}10^6 d^5$$

From (9.42) with τ≤120MPa and re-arranging for *D*

$$D = \frac{\tau\pi d^3}{8W} = \frac{120\text{x}10^6 \pi d^3}{8\text{x}35} = 1.346\text{x}10^6 d^3$$

Eliminating *D* from these two equations we have

$$d = \sqrt[4]{\frac{45\text{x}10^6}{2.439\text{x}10^{18}}} = 2.07\text{mm}$$

It follows that the mean coil diameter is

$$D = 1.346\text{x}10^6\left(2.07\text{x}10^{-3}\right)^3 = 11.94\text{mm}$$

Finally, the number of coils is equal to

$$n = \frac{l_s}{d} = \frac{50}{2.07} \approx 24$$

9.6 Considering the right hand side of the ring shown in Figure 9.19 then the bending moment, M, at angle θ is seen to be

$$M = \left(R - R\cos\theta\right)P = RP\left(1-\cos\theta\right)$$

For pure bending the strain energy, U, of a beam is given by (9.35) which in the present case is

$$U = \int_0^L \frac{M^2 dx}{2EI} = \int_0^\pi \frac{M^2 R d\theta}{2EI}$$

substituting M we have

$$U = \frac{1}{2EI}\int_0^\pi \left[RP\left(1-\cos\theta\right)\right]^2 R d\theta = \frac{R^3P^2}{2EI}\int_0^\pi \left(1 - 2\cos\theta + \cos^2\theta\right)d\theta$$

Performing the integration

$$U = \frac{R^3P^2}{2EI}\left[\theta - 2\sin\theta + \frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^\pi = \frac{3\pi R^3P^2}{4EI}$$

An application of Castigliano's second theorem, (9.59), gives the displacement, δ_u, at the point at which the point load P acts

$$\delta_u = \frac{\partial U}{\partial P} = \frac{3\pi R^3 P}{2EI}$$

Since the total gap, δ, between the two opposing point forces is equal to twice δ_u then

$$\delta = 2\delta_u = \frac{3\pi R^3 P}{EI}$$

as required.

To determine the value of P required to produce a total gap of δ=10mm we can re-arrange the above expression for P with $I=w^4/12$, where w=2.5mm is the width of the square section

$$P = \frac{EI\delta}{3\pi R^3} = \frac{210\text{x}10^9\,\text{x}39.0625\text{x}10^{-12}\,\text{x}10\text{x}10^{-3}}{3\text{x}(45\text{x}10^{-3})^3} = 95.5\text{N}$$

Inspection of the expression for bending moment M we observe that M obtains a maximum at θ=180° (position perpendicular to the applied forces P, as expected) and is equal to $M_{max}=2RP$=2x45x10^{-3}x95.5=8.595N. Thus, the maximum bending stress is, (5.30)

$$\sigma_{\max} = -\frac{M_{\max}y}{I} = -\frac{8.595\text{x}\left(\frac{2.5\text{x}10^{-3}}{2}\right)}{39.0625\text{x}10^{-12}} = -275\text{MPa}$$

which is less in magnitude than the tensile yield stress of σ_Y=300MPa.

9.7 From the given beam deflection equation the maximum static deflection, δ_{st}, occurring at $x=L$ is

$$\delta_{st} = \frac{WL^3}{3EI} = \frac{1\text{x}g\text{x}1^3}{3\text{x}210\text{x}10^9\,\text{x}6.75\text{x}10^{-8}} = 0.2307\text{mm}$$

The impact factor Φ is, (9.89)

$$\Phi = 1 + \sqrt{1 + \frac{2h}{\delta_{st}}} = 1 + \sqrt{1 + \frac{2\text{x}0.5}{2.307\text{x}10^{-4}}} = 66.85$$

Therefore, the maximum displacement, δ_{max}, due to the impact of the falling mass is

$$d_{\max} = Fd_{st} = 15.42\,\text{mm}$$

Chapter 10 Solutions

10.1 Refer to §10.2.

10.2 Refer to §10.3.

10.3 Refer to §10.4.

10.4 To determine the slope at the free end of the cantilever beam then we can apply a virtual unit couple moment 1Nm at the free end of the beam, as shown in Figure Sol10.4. The virtual bending moment, M^v, is

$$M^v = 1 \qquad 0 \le x \le L$$

with the bending moment due to the applied load P

$$M = 0 \qquad 0 \le x \le b$$

$$M = P(x-b) \qquad b \le x \le L$$

An analogous equation to (10.36) can be written for the angle of rotation, θ

$$\theta = \frac{1}{EI}\int_L M^v M dx$$

In the present example we have

$$\theta = \frac{1}{EI}\int_b^L P(x-b)dx = \frac{P}{EI}\left[\frac{x^2}{2} - bx\right]_b^L = \frac{P}{EI}\left(\frac{L^2 - 2bL + b^2}{2}\right) = \frac{P}{EI}\frac{(L-b)^2}{2} = \frac{Pa^2}{2EI}$$

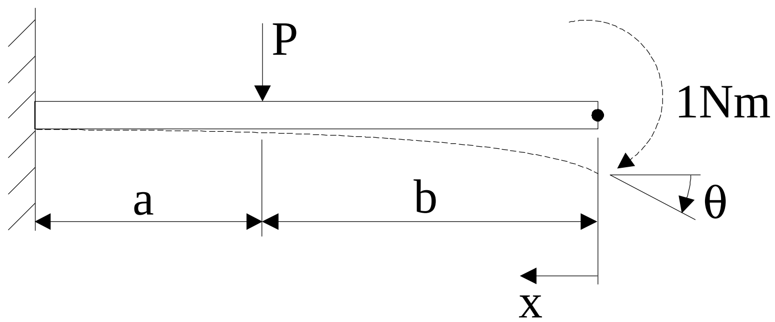


Figure Sol10.4. A cantilever beam with a virtual unit couple moment applied at the free end.

10.5 With a virtual unit load applied at the free end, Figure Sol10.5, then the associated bending moment is

$$M^v = -(R - R\sin\theta)$$

whereas the bending moment due to the real applied point force P is

$$M = -PR\cos\theta$$

Thus, from (10.36) the horizontal deflection, δ, is

$$\delta = \frac{1}{EI}\int_0^{\pi/2} M^v M ds = \frac{1}{EI}\int_0^{\pi/2} -R(1-\sin\theta)(-PR\cos\theta)Rd\theta =$$

$$= \frac{PR^3}{EI}\left[\sin\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2} = \frac{PR^3}{2EI}$$

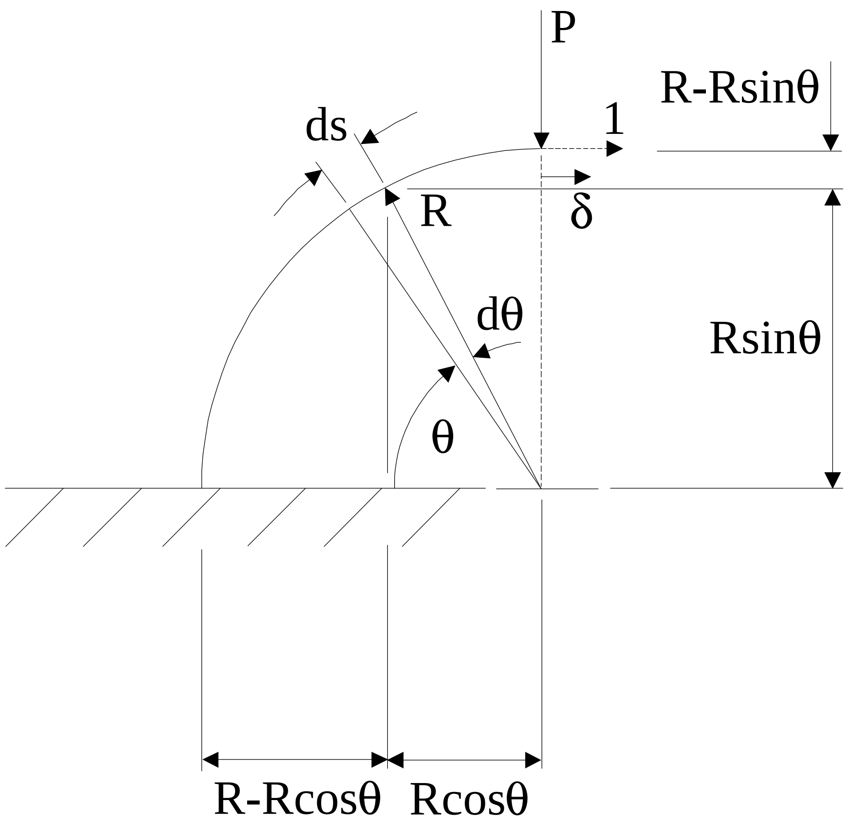


Figure Sol10.5. A quarter circle beam subject to a concentrated force *P* and virtual unit force at the free end.

10.6 To determine the vertical deflection at the free end of the beam we add a virtual unit load at the free end, see Figure Sol10.6. For the virtual unit load the bending moment is

$$M^v = x \qquad 0 \le x \le 7$$

For the real applied loading

$$M = 0 \qquad 0 \le x \le x3$$

$$M = 10(x-3) \qquad 3 \le x \le 4$$

$$M = 10(x-3) + 20(x-4) \qquad 4 \le x \le 7$$

From (10.36) we have

$$EId = \int_L M^v M dx = \int_3^4 10(x-3)xdx + \int_4^7 [10(x-3)+20(x-4)]xdx =$$

$$= 10\left[\frac{x^3}{3} - \frac{3x^2}{2}\right]_3^4 + \left[10x^3 - 55x^2\right]_4^7 = 9{,}768.\dot{3}$$

Re-arranging for δ we have

$$d = \frac{9{,}768.\dot{3}}{205\text{x}10^9\left(5\text{x}10^{-3}\right)} = 9.53\text{mm}$$

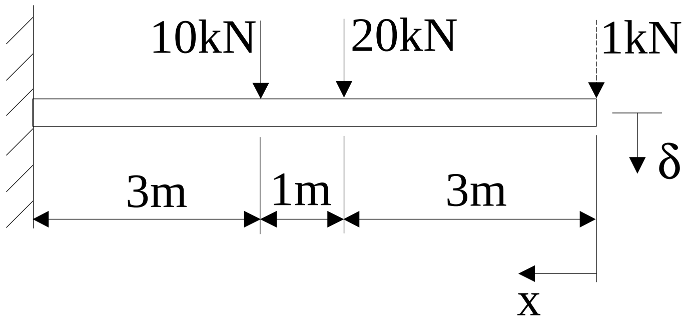


Figure Sol10.6. A cantilever beam subject to real concentrated loads of 10kN and 20kN and a virtual concentrated unit load at the free end of the beam.

10.7 To determine the deflection at the free end of the beam we apply a unit virtual point force at this point. With *x* measured from the free end of the beam then for the virtual unit load the virtual bending moment, M^v, is

$$M^v = x \quad 0 \le x \le 3a$$

For the real applied loading system we have the following bending moments, *M*

$$M = 0 \quad 0 \le x \le a$$

$$M = W(x-a) \quad a \le x \le 2a$$

$$M = W(x-a) + W(x-2a) \quad 2a \le x \le 3a$$

From (10.36) the displacement, δ, at the free end is

$$EId = \int_a^{2a} W(x-a)x dx + \int_{2a}^{3a} \left[W(x-a) + W(x-2a)\right] x dx$$

with no integral in the interval $0 \le x \le a$ because M=0. Performing the integrations we find the desired solution

$$d = \frac{6Wa^3}{EI}$$

Chapter 11 Solutions

11.1 From (11.27) we have

$$\frac{s_{qq} - s_{rr}}{r} = \frac{p}{r}\left[\frac{(b/r)^2+1}{(b/a)^2-1} + \frac{(b/r)^2-1}{(b/a)^2-1}\right] = \frac{2p}{r}\left[\frac{(b/r)^2}{(b/a)^2-1}\right]$$

Differentiating the radial stress with respect to r we have

$$\frac{\P s_{rr}}{\P r} = -\frac{p}{(b/a)^2-1}\left(-2b^2/r^3\right) = \frac{2p}{r}\left[\frac{(b/r)^2}{(b/a)^2-1}\right]$$

which is equivalent to the above equation and therefore satisfies the equilibrium equation (11.3).

11.2 With a=75mm, b=250mm and p=75MPa then from (11.27) the radial and circumferential stresses on the inner surface, r=75mm, are

$$s_{rr} = -p\left[\frac{(b/r)^2-1}{(b/a)^2-1}\right] = -75\left[\frac{(250/75)^2-1}{(250/75)^2-1}\right] = -75\text{MPa}$$

$$s_{qq} = p\left[\frac{(b/r)^2+1}{(b/a)^2-1}\right] = 75\left[\frac{(250/75)^2+1}{(250/75)^2-1}\right] = 90\text{MPa}$$

Assuming closed ends then from (11.33) the axial stress is

$$s_{zz} = \frac{p}{(b/a)^2-1} = \frac{75}{(250/75)^2-1} = 7.4\text{MPa}$$

11.3 With a=0.5m, b=1m, p_i=5MPa, p_o=100kPa=0.1MPa then the constants A and B in (11.20) are

$$A = \frac{(b/a)^2 p_o - p_i}{1-(b/a)^2} = \frac{(1/0.5)^2\,\text{x}0.1\text{x}10^6 - 5\text{x}10^6}{1-(1/0.5)^2} = 1.5\dot{3}\text{x}10^6$$

$$B = \frac{b^2(p_o - p_i)}{1-(b/a)^2} = \frac{1^2\left(0.1\text{x}10^6 - 5\text{x}10^6\right)}{1-(1/0.5)^2} = 1.6\dot{3}\text{x}10^6$$

From (11.21) the radial and circumferential stresses at the radius r=0.75m are

$$s_{rr} = A - \frac{B}{r^2} = 1.5\dot{3}\text{x}10^6 - \frac{1.6\dot{3}\text{x}10^6}{0.75^2} == -1.37\text{MPa}$$

$$s_{qq} = A + \frac{B}{r^2} = 1.5\dot{3}\text{x}10^6 + \frac{1.6\dot{3}\text{x}10^6}{0.75^2} = 4.44\text{MPa}$$

From (11.23) the axial stress is

$$s_{zz} = \frac{p_o(b/a)^2 - p_i}{1-(b/a)^2} = \frac{0.1(1/0.5)^2 - 5}{1-(1/0.5)^2} = 1.5\dot{3}\text{MPa}$$

11.4 Consider the first vessel with the boundary conditions

$$s_{rr} = -45\text{MPa} \text{ at } r = 75\text{mm}$$

$$s_{rr} = 0 \text{ at } r = 100\text{mm}$$

From Lame's equations, (11.16), we find the constants A and B to be given by

$$A = 5.86\text{x}10^6 \quad , \quad B = 578.6\text{x}10^3$$

For the inner surface, in which $\sigma_{\theta\theta}$ will be maximum, then the circumferential stress is

$$\sigma_{\theta\theta} = A + \frac{B}{r^2} = 57.86\text{x}10^6 + \frac{578.6\text{x}10^6}{\left(75\text{x}10^{-3}\right)^2} = 160.72\text{MPa}$$

For a safety factor of 2 then the maximum allowable cylindrical stress for the second cylindrical pressure vessel will be 160.72/2=80.36MPa.

For the second pressure vessel then our boundary condition (σ_{rr}=0 at the inner surface) combined with the maximum design circumferential stress give the two simultaneous equations, (11.16)

$$\sigma_{\theta\theta} = A' + \frac{B'}{r_i^2} \quad \Rightarrow \quad 80.36\text{x}10^6 = A' + \frac{B'}{\left(75\text{x}10^{-3}\right)^2}$$

$$\sigma_{rr} = A' - \frac{B'}{r^2} \quad \Rightarrow \quad 0 = A' - \frac{B'}{\left(75\text{x}10^{-3} + 50\text{x}10^{-3}\right)^2}$$

noting the two new constants *A´* and *B´*. Solving for *A´* and *B´* we have

$$A' = 21.27\text{x}10^6 \quad , \quad B' = 332.34\text{x}10^3$$

Re-arranging the radial stress component of Lame's equations (11.16) for applied internal pressure *p* then we have at the inner surface, *r*=75mm

$$p = -\left[A' - \frac{B'}{r^2}\right] = -\left[21.27x10^6 - \frac{332.34x10^3}{\left(75x10^{-3}\right)^2}\right] = 37.8\text{MPa} = 378\text{bar}$$

Thus, the maximum safe working pressure for the second pressure vessel is 378bar.

11.5 We first need to determine the interference pressure, *p*, so that the maximum stress ($\sigma_{\theta\theta}$) at the sleeve-collar interface does not exceed 300x10^6. Therefore, from Lame's equations

$$\sigma_{\theta\theta}\left(R_{ci}\right) = 300\text{x}10^6 = A + \frac{B}{\left(49.5\text{x}10^{-3}\right)^2} \quad , \quad \sigma_{rr}\left(R_{co}\right) = 0 = A - \frac{B}{\left(100\text{x}10^{-3}\right)^2}$$

Solving for *A* and *B* we find

$$A = 59\text{x}10^6 \quad , \quad B = 59\text{x}10^4$$

The radial stress at r=R_{ci} is

$$\sigma_{rr}(R_{ci}) = A - \frac{B}{R_{ci}^2} = 59\text{x}10^6 - \frac{59\text{x}10^4}{\left(49.5\text{x}10^{-3}\right)^2} = -182\text{MPa}$$

Therefore, the interference pressure is 182MPa.

From (11.40) the radial compression on the shaft is

$$u_s = -\frac{pR_s}{E}\left(1-\nu\right) = -\frac{182\text{x}10^6(50\text{x}10^{-3})}{210\text{x}10^9}\left(1-0.3\right) = -30.\dot{3}\text{x}10^{-6}\,\text{m}$$

From (11.44) the radial expansion of the collar is

$$u_c = \frac{R_{ci}}{E}\left[\frac{pR_{ci}^2}{R_{co}^2 - R_{ci}^2}\right]\left[1-\nu + \left(1+\nu\right)\left(\frac{R_{co}}{R_{ci}}\right)^2\right]$$

$$= \frac{49.5\text{x}10^{-3}}{210\text{x}10^9}\left[\frac{182\text{x}10^6(49.5\text{x}10^{-3})^2}{(100\text{x}10^{-3})^2 - (49.5\text{x}10^{-3})^2}\right]\left[1-0.2+(1+0.3)\left(\frac{100}{49.5}\right)^2\right] = 85\text{x}10^{-6}\,\text{m}$$

Therefore, the total radial interference is, (11.45)

$$\delta = \|u_c\| + \|u_s\| = 115.\dot{3}\text{x}10^{-6}\,\text{m}$$

11.6 Let the inner and outer cylinders be denoted by vessels 1 and 2 as in §11.10. From (11.52), (11.51), (11.48) and (11.27) the total radial interference is given by

$$\delta = u_2 - u_1 = \frac{b}{E}\left[\left(\sigma_{\theta\theta,2} - \nu\sigma_{rr,2}\right) - \left(\sigma_{\theta\theta,1} - \nu\sigma_{rr,1}\right)\right] = \frac{1}{Eb}\left[b^2(1-\nu)(A_2 - A_1) + (1+\nu)(B_2 - B_1)\right]$$

where

$$A_2 - A_1 = \frac{pb^2\left(a^2 - c^2\right)}{\left(a^2 - b^2\right)\left(c^2 - b^2\right)} \quad , \quad B_2 - B_1 = \frac{pb^4\left(a^2 - c^2\right)}{\left(a^2 - b^2\right)\left(c^2 - b^2\right)} = b^2\left(A_2 - A_1\right)$$

Substituting (A_2-A_1) and (B_2-B_1) into δ we finally arrive at

$$\delta = \frac{b(A_2 - A_1)}{E} = \frac{pb^3}{E}\frac{\left(a^2 - c^2\right)}{\left(a^2 - b^2\right)\left(c^2 - b^2\right)}$$

Re-arranging for *p* then the interference pressure is

$$p = \frac{E\delta}{b^3}\frac{\left(a^2 - b^2\right)\left(c^2 - b^2\right)}{\left(a^2 - c^2\right)} = \frac{210\text{x}10^9\left(100\text{x}10^{-6}\right)}{\left(75\text{x}10^{-3}\right)^3}\left[\frac{\left(50^2 - 75^2\right)\left(100^2 - 75^2\right)}{\left(50^2 - 100^2\right)}\right]\text{x}10^{-6} = 91\text{MPa}$$

11.7 From (11.72) with p=100MPa, a=100mm, b=175mm and r=100mm then the circumferential stress is

$$\sigma_{\theta\theta} = \frac{p}{2}\left[\frac{2 + (b/r)^3}{(b/a)^3 - 1}\right] = \frac{100}{2}\left[\frac{2 + (175/100)^3}{(175/100)^3 - 1}\right] = 68.66\text{MPa}$$

Chapter 12 Solutions

12.1 Letting the total strain, ε, be the sum of the elastic, ε_e, and plastic, ε_p, strains and with $\varepsilon_p=\varepsilon_e/5$ then ε is

$$\varepsilon = \varepsilon_e + \varepsilon_p = \varepsilon_e + \frac{\varepsilon_e}{5} = \frac{6\varepsilon_e}{5}$$

From Hooke's law $\varepsilon_e=\sigma_e/E$ and at the point of yielding then $\sigma_e=\sigma_Y$ and the total strain is

$$\varepsilon = \frac{6\sigma_Y}{5E}$$

Substituting this total strain into the constitutive equation

$$\sigma_Y = \frac{E}{200}\left(\frac{6\sigma_Y}{5E}\right)^{1/5}$$

Solving for σ_Y then we arrive at

$$\sigma_Y = 1.39\text{x}10^{-3}E = \frac{E}{719}$$

12.2 From (12.9) the mean yield stress, σ_m, is

$$\sigma_m = \frac{1}{\varepsilon}\int \sigma_Y d\varepsilon = \frac{1}{\varepsilon}\int_0^{\varepsilon}\left[\sigma_Y + B\left(\varepsilon - \frac{\sigma_Y}{E}\right)\right]d\varepsilon = \sigma_Y + B\left(\frac{\varepsilon}{2} - \frac{\sigma_Y}{E}\right)$$

12.3 To determine the empirical constants C and n of Ludwik's power law from the given engineering stress and strain data then we require relations (12.11) and (12.14) which relate the true and engineering components, that is

$$\sigma = \sigma_0(1+e_0) = 340(1+0.3) = 442\,\text{MPa}$$

$$\varepsilon = \ln(1+e_0) = \ln(1+0.3) = 0.2624$$

Inserting these into the Ludwik power law, (12.1), we have

$$442 = C(0.2624)^n$$

At the point of plastic instability we know from (12.15) that $\sigma=d\sigma/d\varepsilon$, or from Example 12.2 that $\sigma=Cn^n$ and $\varepsilon=n$ for Ludwik's power law. Therefore, $n=\varepsilon=0.2624$ and upon substitution into the previous equation and solving for C we have

$$442 = C(0.2624)^{0.2624} \quad \Rightarrow \quad C = 627.9$$

12.4 The maximum bending stress, σ, and shear stress, τ_{max}, for the circular cross-section bar are, (5.30) and (3.11)

$$\sigma = \frac{My}{I} = \frac{M(d/2)}{I} = \frac{32M}{\pi d^3} \quad , \quad \tau_{\max} = \frac{TR}{J} = \frac{Td}{2J} = \frac{16T}{\pi d^3}$$

and substituting into (12.61)

$$\left(\frac{32M}{\pi d^3}\right)^2 + 3\left(\frac{16T}{\pi d^3}\right)^2 = \sigma_Y^2$$

Substituting $M=cT$ and dividing by τ_{max} we arrive at the required result

$$\frac{\sigma_Y}{\tau_{\max}} = \sqrt{3+4c^2}$$

12.5 Tresca's yield criterion is given by (12.29) with the difference in principal stresses obtained from (12.31). Therefore, the yield stress is

$$s_Y = \sqrt{\left(s_{xx} - s_{yy}\right)^2 + 4t_{xy}^2} = \sqrt{\left(500-100\right)^2 + 4\left(100\right)^2} = 447\text{MPa}$$

To determine the value of the yield stress according to the Huber-von Mises yield criterion we will first evaluate the principal stresses, (12.30)

$$s_{1,2} = \left(\frac{s_{xx} + s_{yy}}{2}\right) \pm \sqrt{\left(\frac{s_{xx} - s_{yy}}{2}\right)^2 + t_{xy}^2}$$

$$= \left(\frac{500+100}{2}\right) \pm \sqrt{\left(\frac{500-100}{2}\right)^2 + 100^2} = 523\text{MPa}\ ,\ 76\text{MPa}$$

From (12.46) the yield stress is

$$s_Y = \sqrt{s_1^2 - s_1 s_2 + s_2^2} = \sqrt{523^2 - \left(523\right)\left(76\right) + 76^2} = 489\text{MPa}$$

12.6 From (12.66) the total torque, T, consisting of the elastic torque, T_E, and plastic torque, T_P, is

$$T = T_E + T_P = \frac{2}{3}pkR^3\left[1 - \frac{1}{4}\left(\frac{R_p}{R}\right)^3\right] = \frac{2}{3}p\text{x}175\text{x}10^6\left(25\text{x}10^{-3}\right)^3\left[1 - \frac{1}{4}\left(\frac{16}{25}\right)^3\right] = 5.352\text{kNm}$$

with R=50/2=25mm and R_p=25-9=16mm. At first yield R_p=R and the torque is equal to

$$T_Y = \frac{p}{2}kR^3 = 4.295\text{kNm}$$

and when the entire section is fully plastic then R_p=0 and the torque is equal to

$$T_{FP} = \frac{2}{3}pkR^3 = 5.727\text{kNm}$$

12.7 From (12.90) the applied bending moment, M_Y, at first yield is

$$M_Y = \frac{bh^2 s_Y}{6} = \frac{2.5\text{x}10^{-2}\left(4\text{x}10^{-2}\right)^2\text{x}250\text{x}10^6}{6} = 1.6\dot{6}\,\text{kNm}$$

From (12.96) the value of applied moment, M, to cause the plasticity to extend to a depth of 1cm is

$$M = \frac{3M_Y}{2}\left[1 - \frac{1}{3}\left(\frac{y_0}{h/2}\right)^2\right] = \frac{3\left(1.\dot{6}\,\text{x}10^3\right)}{2}\left[1 - \frac{1}{3}\left(\frac{1\text{x}10^{-2}}{2\text{x}10^{-2}}\right)^2\right] = 2.29\text{kNm}$$

For the plasticity to spread throughout the entire cross-section then y_0=0 at which point the bending moment is

$$M_{FP} = \frac{3M_Y}{2} = \frac{3\left(1.\dot{6}\,\text{x}10^3\right)}{2} = 2.5\text{kNm}$$

Chapter 13 Solutions

13.1 At the nodes *i* and *j* we have, (13.23)

$$\phi_i = \alpha_1 + \alpha_2 x_i \quad , \quad \phi_j = \alpha_1 + \alpha_2 x_j$$

Solving for α_1 and α_2 we find

$$\alpha_1 = \frac{\phi_i x_j - \phi_j x_i}{L} \quad , \quad \alpha_2 = \frac{\phi_j - \phi_i}{L}$$

and substituting into the interpolation function (13.23) then ϕ is

$$\phi = \frac{\phi_i x_j - \phi_j x_i}{L} + \frac{\left(\phi_j - \phi_i\right)x}{L} = \left(\frac{x_j - x}{L}\right)\phi_i + \left(\frac{x - x_i}{L}\right)\phi_j = N_i \phi_i + N_j \phi_j$$

where N_i and N_j are the shape functions of the element and have the following properties:

- N_i=1 at x=x_i and N_i=0 at x=x_j.
- N_j=1 at x=x_j and N_j=0 at x=x_i.
- The sum of N_i and N_j is always equal to unity for x within the range $x_i \le x \le x_j$.
- The shape functions are of the same order as the interpolation function.

With x_i=2 and x_j=6 then the L=x_j-x_i=4. With ϕ_i=10 at x_i=2 and ϕ_j=20 at x_j=6 then from the above interpolation function at x=3 we have N_i=3/4 and N_j=1/4 with ϕ=12.5. The value of ϕ at x=3 is seen to be a linear interpolation of the nodal values.

13.2 From (13.33) the **D** matrix for plane stress is

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} = 75.95\text{x}10^9 \begin{bmatrix} 1 & 0.28 & 0 \\ 0.28 & 1 & 0 \\ 0 & 0 & 0.36 \end{bmatrix}$$

and from (13.36) the **D** matrix for plane strain is

$$[D] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} = 124.29\text{x}10^9 \begin{bmatrix} 0.72 & 0.28 & 0 \\ 0.28 & 0.72 & 0 \\ 0 & 0 & 0.22 \end{bmatrix}$$

From (13.32) the stress vector for plane stress is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = 75.95\text{x}10^9 \begin{bmatrix} 1 & 0.28 & 0 \\ 0.28 & 1 & 0 \\ 0 & 0 & 0.36 \end{bmatrix}\begin{Bmatrix} 60 \\ 80 \\ 55 \end{Bmatrix}\text{x}10^{-6} = \begin{Bmatrix} 6.26 \\ 7.35 \\ 1.5 \end{Bmatrix}\text{x}10^6$$

and similarly for plane strain

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = [D]\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = 124.29\text{x}10^9 \begin{bmatrix} 0.72 & 0.28 & 0 \\ 0.28 & 0.721 & 0 \\ 0 & 0 & 0.22 \end{bmatrix}\begin{Bmatrix} 60 \\ 80 \\ 55 \end{Bmatrix}\text{x}10^{-6} = \begin{Bmatrix} 8.15 \\ 9.25 \\ 1.5 \end{Bmatrix}\text{x}10^6$$

13.3 The cross-sectional areas of elements 1 and 2 are

$$A_1 = \frac{\pi D_1^2}{4} = \frac{\pi 80^2}{4} = 5{,}027\text{mm}^2 \quad , \quad A_2 = \frac{\pi D_2^2}{4} = \frac{\pi 50^2}{4} = 1{,}964\text{mm}^2$$

The element stiffness matrices are, (13.44)

$$\left[K^1\right]=\frac{5{,}027\left(200\text{x}10^3\right)}{500}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=2010.8\text{x}10^3\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\text{N / mm}$$

$$\left[K^2\right]=\frac{1964\left(120\text{x}10^3\right)}{700}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=336.7\text{x}10^3\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\text{N / mm}$$

The structure stiffness matrix and force vector are, using node ordering (1,2,3)

$$\left[K\right]_s=\begin{bmatrix}2010.8 & -2010.8 & 0\\ -2010.8 & 201.8+336.6 & -336.7\\ 0 & -336.7 & 336.7\end{bmatrix}\text{N / mm}\ ,\quad \left\{F\right\}_s=\begin{Bmatrix}0\\ 0\\ 185\end{Bmatrix}\text{x}10^3\,\text{N}$$

Incorporating the boundary condition u_1=0 then the structure system of equations to be solved for **U** is

$$10^3\begin{bmatrix}2010.8 & -2010.8 & 0\\ -2010.8 & 2347.5 & -336.7\\ 0 & -336.7 & 336.7\end{bmatrix}\begin{Bmatrix}0\\ u_2\\ u_3\end{Bmatrix}=\begin{Bmatrix}0+R_1\\ 0\\ -185\end{Bmatrix}\text{x}10^3$$

where R_1 is the reaction at node 1. Performing row multiplications we have

$$10^3\left(-2010.8u_2\right)=R_1$$

$$10^3\left(2347.5u_2-336.7u_3\right)=0$$

$$10^3\left(-336.7u_2+336.7u_3\right)=-185\text{x}10^3$$

Solving these equations we find u_1=0, u_2=-0.092mm and u_3=-0.6415mm. Both u_2 and u_3 are negative and $u_3<u_2$ as expected. An additional check illustrates that R_1 is equal and opposite to the applied force of -185kN. From (13.30) the element strains are

$$e^1_{xx}=\frac{1}{L}\left(-u_1+u_2\right)=\frac{1}{500}\left(0-0.092\right)=-184\text{x}10^{-6}$$

$$e^2_{xx}=\frac{1}{L}\left(-u_2+u_3\right)=\frac{1}{700}\left(0.092-0.6415\right)=-785\text{x}10^{-6}$$

From (13.32) the element stresses are

$$s^1_{xx}=E_1e^1_{xx}=200\text{x}10^3\left(-184\text{x}10^{-6}\right)=-36.8\text{N / mm}^2$$

$$s^2_{xx}=E_2e^2_{xx}=120\text{x}10^3\left(-785\text{x}10^{-6}\right)=-94.28\text{N / mm}^2$$

and are found to agree exactly with the theoretical estimates

$$s^1_{xx}=\frac{P}{A_1}=-\frac{185\text{x}10^3}{5{,}027}=-36.8\text{N / mm}^2$$

$$s^2_{xx}=\frac{P}{A_2}=-\frac{185\text{x}10^3}{1{,}964}=-94.28\text{N / mm}^2$$

13.4 The stiffness matrices of elements 1 and 2 are, (13.81)

$$\left[K^1\right]=\begin{bmatrix}0.25 & & & \\ -0.433 & 0.75 & \text{sym.} & \\ -0.25 & 0.433 & 0.25 & \\ 0.433 & -0.75 & -0.433 & 0.75\end{bmatrix}\ ,\quad \left[K^2\right]=\begin{bmatrix}0.25 & & & \\ 0.433 & 0.75 & \text{sym.} & \\ -0.25 & -0.433 & 0.25 & \\ -0.433 & -0.75 & 0.433 & 0.75\end{bmatrix}$$

Inserting into the global stiffness matrix we find the structure system of equations

$$346.41\text{x}10^6\begin{bmatrix} 0.25 & & & & & \\ -0.433 & 0.75 & & & & \\ -0.25 & 0.433 & 0.25 & \text{sym.} & & \\ 0.433 & -0.75 & 0 & 1.5 & & \\ 0 & 0 & -0.25 & -0.433 & 0.25 & \\ 0 & 0 & -0.433 & -0.75 & 0.433 & 0.75 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ u_2 \\ v_2 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1000 \\ 0 \\ 0 \end{Bmatrix}$$

with node ordering 1, 2 and 3. Multiplying rows 3 and 4 we find

$$0.25u_2 = 0$$

$$346.41\text{x}10^6(1.5v_2) = -1000$$

leading to u_2=0 and v_2=-1.9245x10^{-6}m; with u_2=0 due to symmetry. Similarly, the axial strain, stress and force components in elements 1 and 2 are equivalent and so we will consider only element 1. The axial strain is, from (13.83)

$$\{e^1\} = \frac{1}{L_1}[-C \quad -S \quad C \quad S]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$= \frac{1}{1.1547}[\text{-0.5} \quad 0.866 \quad 0.5 \quad \text{-0.866}]\begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1.9245 \end{Bmatrix}\text{x}10^{-6} = 1.4433\text{x}10^{-6}$$

The axial stress and force are

$$s^1 = Ee^1 = 0.2887\text{MPa} \quad , \quad F^1 = s^1 A = 577.32\text{N}$$

From the free-body diagram in Figure 13.28 we observe that 2Tcos30°=1kN and therefore T=577.35N which agrees exactly with the finite element prediction. The displacement v_2 is found by an application of Castigliano’s second theorem as in Exercise 9.4

$$v_2 = -\frac{Pd}{2EA\cos^3 J} = -1.9245\text{x}10^{-6}\,\text{m}$$

which, again, is exact agreement with the finite element estimate.

13.5 The cross-sectional area, A, and perimeter, P, for both elements are 25x10^{-6}m^2 and 20x10^{-3}m respectively. Letting elements 1 and 2 have nodes (1,2) and (2,3) respectively then the stiffness matrix for element 1 has contributions due to conduction and perimeter convection, (13.118)

$$[K^1] = \frac{k_{xx}A}{L_1}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{hPL_1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5.16 & -4.9167 \\ -4.9167 & 5.16 \end{bmatrix}\text{x}10^{-3}$$

with the force vector due to $\{F_h\}$ only, (13.119)

$$\{F^1\} = \frac{hT_\infty PL_1}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Element 2 is identical to element 1 except that it also experiences end-convection through node 2 so that the stiffness matrix of element 3 is, (13.118)

$$\left[K^3\right]=\begin{bmatrix}5.16 & -4.9167\\ -4.9167 & 5.16\end{bmatrix}\text{x}10^{-3}+hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}5.16 & -4.9167\\ -4.9167 & 5.185\end{bmatrix}\text{x}10^{-3}$$

Similarly, adding the end-convection term to the force vector of element 2 we have

$$\left\{F^3\right\}=\begin{Bmatrix}0\\0\end{Bmatrix}+hT_\infty A\begin{Bmatrix}0\\1\end{Bmatrix}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Assembling the element contributions into the structure stiffness matrix and force vector we find

$$10^{-3}\begin{bmatrix}5.16 & -4.9167 & 0\\ -4.9167 & 5.16+5.16 & -4.9167\\ 0 & -4.9167 & 5.185\end{bmatrix}\begin{Bmatrix}T_1\\T_2\\T_3\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

with node ordering 1 to 3. We have a prescribed temperature of 100°C at node 1 which results in a non-homogeneous boundary condition. The stiffness matrix and force vector are modified by first setting all non-diagonal terms in the first row and column of the stiffness matrix to zero. Also, the term (-4.9167)x100°C=-491.67 on the left hand side of the second equation is transposed to the right hand side as +491.67. The resulting systems of equations is now given by

$$10^{-3}\begin{bmatrix}1 & 0 & 0\\ 0 & 10.32 & -4.9167\\ 0 & -4.9167 & 5.185\end{bmatrix}\begin{Bmatrix}T_1\\T_2\\T_3\end{Bmatrix}=\begin{Bmatrix}100\\491.67\\0\end{Bmatrix}$$

The second through to third equations are now solved in the usual manner, with the solution vector given by $\{T\}$={100,86.87,82.34}.

13.6 All three elements experience no perimeter convection (P=0) with element 3 experiencing convection at node 4. Assuming a unit cross-sectional area for all three elements then the stiffness matrix and force vector for element 1 are, (13.118) and (13.119)

$$\left[K^1\right]=\frac{k_{xx}A}{L_1}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=0.1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\quad,\quad\left\{F^1\right\}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

and similarly for element 2

$$\left[K^2\right]=\frac{k_{xx}A}{L_2}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}=0.1\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\quad,\quad\left\{F^2\right\}=\begin{Bmatrix}0\\0\end{Bmatrix}$$

Element 3 consists of an additional end-convection term

$$\left[K^3\right]=\frac{k_{xx}A}{L_3}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}+hA\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0.1 & -0.1\\ -0.1 & 50.1\end{bmatrix}\quad,\quad\left\{F^3\right\}=hT_\infty A\begin{Bmatrix}0\\1\end{Bmatrix}=\begin{Bmatrix}0\\-250\end{Bmatrix}$$

Assembling the elements into the structure stiffness matrix and force vector gives, with node ordering 1 to 4

$$\begin{bmatrix}0.1 & -0.1 & 0 & 0\\ -0.1 & 0.2 & -0.1 & 0\\ 0 & -0.1 & 0.2 & -0.1\\ 0 & 0 & -0.1 & 50.1\end{bmatrix}\begin{Bmatrix}T_1\\T_2\\T_3\\T_4\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\-250\end{Bmatrix}$$

Incorporating the prescribed boundary condition T_1=25°C then the system of equations is modified as follows

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2 & -0.1 & 0 \\ 0 & -0.1 & 0.2 & -0.1 \\ 0 & 0 & -0.1 & 50.1 \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} 21 \\ 2.5 \\ 0 \\ -250 \end{Bmatrix}$$

where the right hand side of the first equation is set to 25°C. The term (-0.1)x25°C=-2.5 on the left hand side of the second equation is transposed to the right hand side as +2.5. Solution of the system of equations yields {T}={25,15,5,-5}. From Fourier's law the heat flux for an element of length L and nodes i and j is

$$q_x = -k_{xx}\frac{dT}{dx} = k_{xx}[B]\{T\} = -k_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} T_i \\ T_j \end{Bmatrix}$$

Evaluation of q_x for all three elements reveals that the heat flux is constant and equal to $Q_x = q_x A = q_x = 1$ for all three elements.

13.7 The stiffness matrices for all three elements are equivalent and given by (13.136)

$$[K^1] = [K^2] = [K^3] = \frac{Ak_{xx}}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{(3.1416\text{x}10^{02})1\text{x}10^{-2}}{0.3\dot{3}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 9.425\text{x}10^{-4}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Since there are no sources or sinks and no applied surface flow rates then both Q and q are equal to zero in (13.138) so that the element force vectors are

$$\{F^1\} = \{F^2\} = \{F^3\} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assembling the element components we have the following structure system of equations for unknown fluid heads p_1, …, p_4

$$9.425x10^{-4}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Incorporating the non-homogeneous boundary conditions of p_1=0.2m and p_4=0.1m in a similar manner to that discussed in Example 13.4 we arrive at

$$9.425\text{x}10^{-4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & \end{bmatrix}\begin{Bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 1.885\text{x}10^{-4} \\ 1.885\text{x}10^{-4} \\ 9.425\text{x}10^{-5} \\ 9.425\text{x}10^{-5} \end{Bmatrix}$$

Solving the second and third equations for p_2 and p_3 then the solution vector is {P}={0.2,0.16,0.13,0.2}. From (13.130) the element velocity for an element of length L with nodes i and j is

$$v_x = -k_{xx}\{g\} = -k_{xx}[B]\{P\} = -k_{xx}\begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}\begin{Bmatrix} p_i \\ p_j \end{Bmatrix}$$

For example, for element 1 we have

$$v_x = -1\text{x}10^{-2}\begin{bmatrix} -\frac{1}{0.3\dot{3}} & \frac{1}{0.3\dot{3}} \end{bmatrix}\begin{Bmatrix} 0.2 \\ 0.1\dot{6} \end{Bmatrix} = 1\text{x}10^{-3}\,\text{m/s}$$

with equivalent velocities for elements 2 and 3.

Chapter 14 Solutions

14.1 From §1.11 the in-plane strains are

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = 2axy^2 \quad , \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} = bx^3 \quad , \quad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(2ax^2y + 3bx^2y\right)$$

Differentiating the strains

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} = \frac{\partial}{\partial y}\left(4axy\right) = 4ax \quad , \quad \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial}{\partial x}\left(3bx^2\right) = 6bx$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial}{\partial y}\left[\frac{1}{2}\left(4axy + 6bxy\right)\right] = \frac{1}{2}\left(4ax + 6bx\right)$$

Upon substituting these into the compatibility equation, (14.21), we observe that u and v are compatible.

14.2 Differentiating ϕ we find

$$\frac{\partial \phi}{\partial x} = 2Ax + By \quad , \quad \frac{\partial^2 \phi}{\partial x^2} = 2A \quad , \quad \frac{\partial^3 \phi}{\partial x^3} = 0 \quad , \quad \cdots$$

$$\frac{\partial \phi}{\partial y} = 2Cy + Bx \quad , \quad \frac{\partial^2 \phi}{\partial y^2} = 2C \quad , \quad \frac{\partial^3 \phi}{\partial y^3} = 0 \quad , \quad \cdots$$

$$\frac{\partial^2 \phi}{\partial x^2 \partial y} = 0 \quad , \quad \cdots$$

Thus, we observe that ϕ satisfies $\nabla^4\phi=0$ since all terms of ϕ are less than power 4.

The stresses follow immediately from the Airy stresses, (14.46)

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \frac{\partial}{\partial y}\left(Bx + 2Cy\right) = 2C$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(2Ax + Bx\right) = 2A$$

$$\tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\frac{\partial}{\partial x}\left(2Ax + By\right) = -B$$

It follows that ϕ provides a solution for a plate subject to uniform stresses along its sides of $\sigma_{xx}=2C$, $\sigma_{yy}=2A$ and $\tau_{xy}=-B$.

From the Hookian equations for a state of plane stress, (14.42), the in-plane strains are

$$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] = \frac{2}{E}\left(C - \nu A\right)$$

$$\varepsilon_{yy} = \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] = \frac{2}{E}\left(A - \nu C\right)$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = -\frac{2(1+\nu)}{E}B$$

Integrating the strains we find the displacements

$$u = \int \varepsilon_{xx}dx = \frac{2}{E}\left(C - \nu A\right)dx = \frac{2}{E}\left(C - \nu A\right)x + f(y)$$

$$v = \int \varepsilon_{yy}dy = \frac{2}{E}\left(A - \nu C\right)dy = \frac{2}{E}\left(A - \nu C\right)y + g(x)$$

where the functions $f(y)$ and $g(x)$ are determined from the boundary conditions.

14.3 Let the resultant stress be $S(S_x,S_y,S_z)$ with

$$S^2 = S_x^2 + S_y^2 + S_z^2$$

This resultant stress consists of both normal, S_N, and shear, S_S, components

$$S^2 = S_N^2 + S_S^2$$

If the direction cosines of ABC are $l=\cos\alpha$, $m=\cos\beta$ and $n=\cos\gamma$ then

$$S_N = lS_x + mS_y + nS_z$$

where (S_x,S_y,S_z) in terms of the coordinate components are

$$S_x = l\sigma_{xx} + m\tau_{xy} + n\tau_{xz}$$

$$S_y = l\tau_{yx} + m\sigma_{yy} + n\tau_{yz}$$

$$S_z = l\tau_{zx} + m\tau_{zy} + n\sigma_{zz}$$

Substituting S_x, S_y and S_z into S_N we have

$$S_N = l^2\sigma_{xx} + m^2\sigma_{yy} + n^2\sigma_{zz} + 2\left(lm\tau_{xy} + mn\tau_{yz} + ln\tau_{zx}\right) \quad , \quad S_S = \sqrt{S^2 - S_N^2}$$

Substituting σ_{ij} and (l,m,n) into S_x, S_y and S_z we have

$$S_x = \frac{1}{\sqrt{3}}(10+5+15) = \frac{30}{\sqrt{3}} = 17.32$$

$$S_y = \frac{1}{\sqrt{3}}(5+20+25) = \frac{50}{\sqrt{3}} = 28.87$$

$$S_z = \frac{1}{\sqrt{3}}(15+25+30) = \frac{70}{\sqrt{3}} = 40.41$$

with S equal to

$$S = \sqrt{\left(\frac{30}{\sqrt{3}}\right)^2 + \left(\frac{50}{\sqrt{3}}\right)^2 + \left(\frac{70}{\sqrt{3}}\right)^2} = 52.6$$

The normal and shear stresses are

$$S_N = \frac{1}{\sqrt{3}}\left(\frac{30}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\left(\frac{50}{\sqrt{3}}\right) + \frac{1}{\sqrt{3}}\left(\frac{70}{\sqrt{3}}\right) = 50 \quad , \quad S_S = \sqrt{52.6^2 - 50^2} = 16.33$$

The direction of S_N acts normal to plane ABC and is therefore defined by $(l,m,n)=(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})$ with $\alpha=\beta=\gamma=\cos^{-1}(1/\sqrt{3})=54.74°$. A useful check is to ensure that $l^2+m^2+n^2=1$ which is the case. The direction of S_S acts parallel to the plane ABC and let it be defined by (l_s,m_s,n_s) where $l_s=\cos\alpha_s$, $m_s=\cos\beta_s$ and $n_s=\cos\gamma_s$. The components (S_x,S_y,S_z) can now alternatively be defined as, resolving S_N and S_S

$$S_x = S_N\cos\alpha + S_S\cos\alpha_s = lS_N + l_sS_S$$

$$S_y = S_N\cos\beta + S_S\cos\beta_s = mS_N + m_sS_S$$

$$S_z = S_N\cos\gamma + S_S\cos\gamma_s = nS_N + n_sS_S$$

from which it follows

$$l_s = \left(S_x - lS_N\right)/S_S = -\frac{1}{\sqrt{2}}$$

$$m_s = \left(S_y - mS_N\right)/S_S = 0$$

$$n_s = \left(S_z - nS_N\right)/S_S = \frac{1}{\sqrt{2}}$$

confirming that $l_s^2 + m_s^2 + n_s^2 = 1$. The angles α_s, β_s and γ_s are $\alpha_s=\cos^{-1}(-1/\surd 2)=135°$, $\beta_s=\cos^{-1}(0)=90°$ and $\gamma_s=\cos^{-1}(1/\surd 2)=45°$. Finally, we can check that the two direction vectors of S_N and S_S are orthogonal by ensuring that the dot product of (l,m,n) and (l_s,m_s,n_s) is equal to zero, that is $(1/\surd 3,1/\surd 3,1/\surd 3).(-1/\surd 2,0,1/\surd 2)=0$.

14.4 From (14.100) the **C** matrix is

$$[C] = \frac{210\text{x}10^9}{(1+0.3)(1-2\text{x}0.3)}\begin{bmatrix} 1-0.3 & 0.3 & 0 \\ 0.3 & 1-0.3 & 0 \\ 0 & 0 & (1-2\text{x}0.3)/2 \end{bmatrix} = 404\text{x}10^9\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}$$

With $[\varepsilon]^T=[-19,64,3]\text{x}10^{-6}$ then the stress vector is

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{Bmatrix} = 404\text{x}10^9\begin{bmatrix} 0.7 & 0.3 & 0 \\ 0.3 & 0.7 & 0 \\ 0 & 0 & 0.4 \end{bmatrix}\begin{Bmatrix} -19 \\ 64 \\ 3 \end{Bmatrix}\text{x}10^{-6} = \begin{Bmatrix} 2.38 \\ 15.8 \\ 0.48 \end{Bmatrix}\text{MPa}$$

14.5 From (14.190) the shear stresses τ_{xz} and τ_{yz} are

$$\tau_{xz} = \frac{\partial\phi}{\partial y} = -Gqy[1-x] \quad , \quad \tau_{yz} = -\frac{\partial\phi}{\partial x} = \frac{3Gq}{2a}\left[\frac{2ax}{3} - x^2 + y^2\right]$$

The centroid of the triangle is at $(x,y)=(0,0)$ and the three corners are at (2a/3,0), (-a/3, $a/\sqrt{3}$) and (-a/3,- $a/\sqrt{3}$). Substituting these coordinates into τ_{yz} above we observe that τ_{yz} is equal to zero at the centroid and three corners.

14.6 Since the hole in the plate is circular, $a/b=1$, then from (14.212) the stress concentration factor is $K_t=3$. The maximum of σ_P that can be applied to the plate is therefore

$$\sigma_P(\max) = \frac{\sigma_Y}{K_t} = \frac{300\text{MPa}}{3} = 100\text{MPa}$$

From (14.210) the σ_{yy} stress at a distance $x=a/2=6.25$mm from the notch root (notch root radius of $\rho=b^2/a=a=12.5$mm) is

$$\sigma_{yy} = \sigma_P K_t\sqrt{\frac{\rho}{\rho+4x}} = 100\text{x}10^6\text{x}3\sqrt{\frac{12.5}{12.5+4(6.25)}} = 173\text{MPa}$$

14.7 From (7.19) the principal stresses are given by

$$\sigma_{1,2} = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}\right) \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}$$

With the stresses given by (14.223) for the semi-infinite Boussinesq wedge then the three main terms in the above expression are

$$\frac{\sigma_{xx}+\sigma_{yy}}{2} = -\frac{\phi}{\pi r^4}\left(x^3+xy^2\right)$$

$$\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 = \left(\frac{\phi}{\pi r^4}\right)^2\left(xy^2-x^3\right)^2$$

$$\tau_{xy}^2 = \left(\frac{\phi}{\pi r^4}\right)^2 4x^4y^2$$

Noting that the square root term reduces to

$$\left(\frac{s_{xx} - s_{yy}}{2}\right)^2 + t_{xy}^2 = \left(\frac{f}{pr^2}\right)^2 x^2\left(x^2 + y^2\right)$$

and substituting into the principal stresses we finally arrive at the required result

$$s_1 = 0 \quad , \quad s_2 = -\frac{2fx}{pr^2}$$

Chapter 15 Solutions

15.1 Refer to §15.2 for a discussion on equivalent stress and strain.

15.2 Refer to §15.2 for a discussion on the constancy of volume condition and its application to illustrate that Poission's ratio is equal to ½ for incompressible materials.

15.3 For a thin-walled pressure vessel with no shear stresses then the circumferential, axial and radial coordinate stresses $\sigma_{\theta\theta}$(=$pd/2t$), σ_{zz}(=$\sigma_{\theta\theta}/2$) and σ_{rr} are equivalent to the principal stresses σ_1, σ_2 and σ_3; where p is the internal pressure, d is the mean diameter and t is the wall thickness. From (15.1) the equivalent stress is

$$\bar{\sigma} = \frac{1}{\sqrt{2}}\sqrt{\left(\sigma_{\theta\theta} - \sigma_{zz}\right)^2 + \sigma_{zz}^2 + \sigma_{\theta\theta}^2} = \frac{\sqrt{3}}{2}\sigma_{\theta\theta}$$

The equivalent plastic strain is given by (15.2) with the plastic strain increments $d\varepsilon_1^p$, $d\varepsilon_2^p$ and $d\varepsilon_3^p$ determined from the Levy-Mises flow rule. The deviatoric component of $\sigma_{\theta\theta}$ is

$$\sigma'_{\theta\theta} = \sigma_{\theta\theta} - \left(\frac{\sigma_{\theta\theta} + \sigma_{zz} + \sigma_{rr}}{3}\right) = \frac{2}{3}\left[\sigma_{\theta\theta} - \frac{1}{2}\left(\sigma_{zz} + \sigma_{rr}\right)\right]$$

The plastic strain increments are, from (15.22)

$$d\varepsilon_1^p = d\varepsilon_{\theta\theta}^p = \frac{2\lambda}{3}\left[\sigma_{\theta\theta} - \frac{1}{2}\left(\sigma_{zz} + \sigma_{rr}\right)\right] = \frac{\lambda}{2}\sigma_{\theta\theta}$$

$$d\varepsilon_2^p = d\varepsilon_{zz}^p = \frac{2\lambda}{3}\left[\sigma_{zz} - \frac{1}{2}\left(\sigma_{\theta\theta} + \sigma_{rr}\right)\right] = 0$$

$$d\varepsilon_3^p = d\varepsilon_{rr}^p = \frac{2\lambda}{3}\left[\sigma_{rr} - \frac{1}{2}\left(\sigma_{\theta\theta} + \sigma_{zz}\right)\right] = -\frac{\lambda}{2}\sigma_{\theta\theta}$$

with the equivalent plastic strain given by, (15.2)

$$d\bar{\varepsilon}_p = \frac{\sqrt{2}}{3}\sqrt{\left(d\varepsilon_{\theta\theta}^p - d\varepsilon_{zz}^p\right)^2 + \left(d\varepsilon_{zz}^p - d\varepsilon_{rr}^p\right)^2 + \left(d\varepsilon_{\theta\theta}^p - d\varepsilon_{rr}^p\right)^2}$$

15.4 In the absence of shearing stresses, τ_{rz}, then the principal stresses are equal to the coordinate stresses $\sigma_{\theta\theta}$, σ_{rr} and σ_{zz}. With the external pressure p_o=0 and the internal presure p_i=p then we have from Lame's equations, (15.34)

$$\sigma_{rr} = -p$$

$$\sigma_{\theta\theta} = p\left(\frac{k^2+1}{k^2-1}\right) \quad ; \quad k = b/a$$

$$\sigma_{zz} = 0 \quad \text{(open ends)}$$

Let us examine both the Tresca and Huber-von Mises yield criteria starting with Tresca's criterion.

Tresca's Yield Criterion

Since the coordinate stresses are equivalent to the principal stresses, (15.33)

$$\sigma_{\theta\theta} - \sigma_{rr} = \sigma_Y$$

Substituting the radial and circumferential stresses we find that first yield occurs when

$$p = \frac{\sigma_Y}{2}\left(1 - \frac{1}{k^2}\right)$$

Re-arranging for *k*

$$k = \left[\frac{(\sigma_Y / p)}{(\sigma_Y / p) - 2}\right]^{1/2}$$

With p=100MPa and σ_Y=250MPa then $k=\sqrt{5}=2.24$.

Huber-von Mises Yield Criterion

From the Huber-von Mises yield criterion

$$(\sigma_{rr} - \sigma_{\theta\theta})^2 + (\sigma_{\theta\theta} - \sigma_{zz})^2 + (\sigma_{zz} - \sigma_{rr})^2 = 2\sigma_Y^2$$

With σ_{zz}=0 for open ends then this equation reduces to

$$\sigma_{rr}^2 - \sigma_{rr}\sigma_{\theta\theta} + \sigma_{\theta\theta}^2 = \sigma_Y^2$$

Substituting the radial and circumferential stresses and re-arranging for *p* we arrive at

$$p = \sigma_Y \frac{k^2 - 1}{\sqrt{3k^4 + 1}}$$

Re-arranging for *k* we arrive at the following quadratic equation with unknown k^2

$$\left[3p^2 - \sigma_Y^2\right]k^4 + 2\sigma_Y^2 k^2 + \left(p^2 - \sigma_Y^2\right) = 0$$

Solving, then *k* is given by

$$k = \left[\frac{-\sigma_Y^2 \pm \sqrt{\sigma_Y^4 - (3p^2 - \sigma_Y^2)(p^2 - \sigma_Y^2)}}{3p^2 - \sigma_Y^2}\right]^{1/2}$$

With p=100MPa and σ_Y=250MPa then the two solutions of *k* are k=0.69 and k=1.83. Since k>1 then k=1.83 and is approximately 22% less than the Tresca prediction and results in approximately 70% difference in cross-sectional area.

15.5 The required applied internal pressure, *p*, to produce an elastic-plastic boundary to a depth of c=70mm can be found by setting $r=a$ and $p=-\sigma_{rr}$ in (15.40)

$$p = -\sigma_{rr} = \sigma_Y\left[\ln\left(\frac{c}{a}\right) + \frac{1}{2}\left[1 - \left(\frac{c}{b}\right)^2\right]\right] = 300\left[\ln\left(\frac{70}{50}\right) + \frac{1}{2}\left[1 - \left(\frac{70}{100}\right)^2\right]\right] = 177.45\text{MPa}$$

The fully plastic condition is reached when $c=b$ and the required pressure, p_Y, is

$$p_Y = \sigma_Y \ln\left(\frac{b}{a}\right) = 300\ln\left(\frac{100}{50}\right) = 207.94\text{MPa}$$

15.6 From the material power law

$$\frac{d\bar{\sigma}}{d\bar{\varepsilon}^P} = nC\left(\bar{\varepsilon}^P\right)^{n-1}$$

and upon substitution into the plastic instability condition (15.48) we have

$$nC\left(\bar{\varepsilon}^P\right)^{n-1} = \sqrt{3}\bar{\sigma} = \sqrt{3}C\left(\bar{\varepsilon}^P\right)^n \quad \Rightarrow \quad \bar{\varepsilon}^P = \frac{n}{\sqrt{3}} = 0.26$$

From (15.49) the mean radius and wall thickness at the point of plastic instability are

$$r = r_0 e^{\sqrt{3}\bar{\varepsilon}^P/2} = 0.45e^{\sqrt{3}(0.26)/2} = 0.56\text{m} \ , \quad t = t_0 e^{-\sqrt{3}\bar{\varepsilon}^P/2} = 1e^{-\sqrt{3}(0.26)/2} = 0.8\text{mm}$$

15.7 From Exercise 15.6 the equivalent plastic strain remains the same. From (15.57) the mean radius and wall thickness at the point of plastic instability are

$$r = r_0 e^{\bar{e}^P/2} = 0.45e^{0.26/2} = 0.51\text{m} \;\; , \;\; t = t_0 e^{-\bar{e}^P/2} = 1e^{-0.26/2} = 0.88\text{mm}$$

Chapter 16 Solutions

16.1 From (16.5) and (16.6) the constants *a*, *b* and *c* and area *A* are

$$a_i = 8 \ , \ b_i = -2 \ , \ c_i = -1$$
$$a_j = -2 \ , \ b_j = 3 \ , \ c_j = -1$$
$$a_k = -1 \ , \ b_k = -1 \ , \ c_k = 2$$
$$A = 2.5$$

From (16.8) the shape functions at point *p* are

$$N_i = \frac{1}{5} \ , \ N_j = \frac{1}{5} \ , \ N_k = \frac{3}{5}$$

From (16.11) the displacement vector at point *p* is

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix} \begin{Bmatrix} u_i \\ v_i \\ u_j \\ v_j \\ u_k \\ v_k \end{Bmatrix} =$$

$$= \begin{bmatrix} 1/5 & 0 & 1/5 & 0 & 3/5 & 0 \\ 0 & 1/5 & 0 & 1/5 & 0 & 3/5 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 3 \\ 4 \\ 2 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 2 \\ 2.2 \end{Bmatrix}$$

16.2 Denoting nodes (1,2,3) by (*i*,*j*,*k*) to assist in the use of the required formulae then from (16.5) and (16.6) the constants *a*, *b* and *c* and area *A* are

$$a_i = 9 \ , \ b_i = -2 \ , \ c_i = -1$$
$$a_j = -3 \ , \ b_j = 2 \ , \ c_j = -1$$
$$a_k = -2 \ , \ b_k = 0 \ , \ c_k = 2$$
$$A = 2$$

From (16.8) the shape functions are

$$N_i = \frac{1}{4}[9 - 2x - y] \ , \ N_j = \frac{1}{4}[-3 + 2x - y] \ , \ N_k = \frac{1}{4}[-2 + 2y]$$

From (16.11) the [*N*] matrix is

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_k & 0 \\ 0 & N_i & 0 & N_j & 0 & N_k \end{bmatrix}$$

From (16.17) the [*B*] matrix is

$$[B] = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -2 & -1 & 2 & 2 & 0 \end{bmatrix}$$

From (13.33) the [*D*] matrix is

$$[D] = 219x10^3 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \text{N / mm}^2$$

From (16.17) the strain vector is

$$\{e\} = \begin{Bmatrix} e_{xx} \\ e_{yy} \\ g_{xy} \end{Bmatrix} = [B]\{U\} = \frac{1}{4} \begin{bmatrix} -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \\ -1 & -2 & -1 & 2 & 2 & 0 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{Bmatrix} = \begin{Bmatrix} 0.5 \\ 1.5 \\ 2.25 \end{Bmatrix} \text{x}10^{-3}$$

From (16.18) the stress vector is

$$\{s\} = \begin{Bmatrix} s_{xx} \\ s_{yy} \\ t_{xy} \end{Bmatrix} = [D]\{e\} = 220x10^3 \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} \begin{Bmatrix} 0.5 \\ 1.5 \\ 2.25 \end{Bmatrix} \text{x}10^{-3} = \begin{Bmatrix} 209 \\ 363 \\ 173.25 \end{Bmatrix} \text{N / mm}^2$$

16.3 With x_i=3, y_i=3, x_j=4, y_j=1, x_k=5 and y_k=3 then the constants *a*, *b* and *c* and the area, *A*, of the element are

$$a_i = 7\text{x}10^{-6}\,\text{m}^2 \quad , \quad b_i = -2\text{x}10^{-3}\,\text{m} \quad , \quad c_i = 1\text{x}10^{-3}\,\text{m}$$

$$a_j = 6\text{x}10^{-6}\,\text{m}^2 \quad , \quad b_j = 0\text{m} \quad , \quad c_j = -2\text{x}10^{-3}\,\text{m}$$

$$a_k = -9\text{x}10^{-6}\,\text{m}^2 \quad , \quad b_k = 2\text{x}10^{-3}\,\text{m} \quad , \quad c_k = 1\text{x}10^{-3}\,\text{m}$$

$$A = 2\text{x}10^{-6}\,\text{m}^2$$

From (16.17) the [*B*] matrix is

$$[B] = \frac{1}{2A} \begin{bmatrix} b_i & 0 & b_j & 0 & b_k & 0 \\ 0 & c_i & 0 & c_j & 0 & c_k \\ c_i & b_i & c_j & b_j & c_k & b_k \end{bmatrix} = 250 \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & -2 & 0 & 1 & 2 \end{bmatrix}$$

and from (13.33) the [*D*] matrix is

$$[D] = \frac{E}{1-n^2} \begin{bmatrix} 1 & n & 0 \\ n & 1 & 0 \\ 0 & 0 & (1-n)/2 \end{bmatrix} = 2.1978\text{x}10^{11} \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}$$

The stiffness matrix is, (16.34)

$$[K] = [B]^{\text{T}}[D][B]tA =$$

$$= 250 \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \\ 0 & -2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix} (2.1978\text{x}10^{11}) \begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix} 250 \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & -2 & 0 & 1 \\ 1 & 2 & -2 & 0 & 1 & 2 \end{bmatrix} (2\text{x}10^{-3})(2\text{x}10^{-6})$$

Performing the multiplications we find

$$[K]=5.4945\text{x}10^7\begin{bmatrix}4.35 & & & & & \\ 0.1 & 2.4 & & & & \\ 0.7 & 1.4 & 1.4 & \text{sym.} & & \\ 1.2 & -2 & 0 & 4 & & \\ -4.35 & -0.1 & -0.7 & -1.2 & 4.35 & \\ -1.3 & -0.4 & -0.4 & -2 & 1.3 & 2.4\end{bmatrix}$$

noting that the matrix is symmetric ($K_{ij}=K_{ji}$), the principal terms are all positive and non-zero ($K_{ii}>0$) and the sum of all terms in either a row or column are zero ($\sum_{j=1}^{6} K_{ij}=0$).

To determine the force vector we require just the contribution due to the edge pressure. From (16.42) the normal pressure term for edge (i,k) is

$$\{F\}=\{F\}_{pressure}=\frac{L_{ik}t}{2}\begin{Bmatrix}p_x\\ p_y\\ 0\\ 0\\ p_x\\ p_y\end{Bmatrix}=\begin{Bmatrix}0\\ -200\\ 0\\ 0\\ 0\\ -200\end{Bmatrix}$$

16.4 Measuring the length coordinate ξ from node 1 then ξ=3/4 for point p. From (16.60) the shape functions are

$$N_1=2x^2-2x+1=2\left(\frac{3}{4}\right)^2-2\left(\frac{3}{4}\right)+1=-\frac{1}{8}$$

$$N_2=2x^2-x=2\left(\frac{3}{4}\right)^2-\frac{3}{4}=\frac{3}{8}$$

$$N_3=4x-4x^2=4\left(\frac{3}{4}\right)-4\left(\frac{3}{4}\right)^2=\frac{3}{4}$$

The displacement u at point p is, (16.59)

$$u=N_1u_1+N_2u_2+N_3u_3=\left(-\frac{1}{8}\right)2+\left(\frac{3}{8}\right)2.25+\left(\frac{3}{4}\right)2.55=2.51$$

16.5 To determine the element shape functions (16.84) we first require the area coordinates of point p. From (16.6) we find

$$A=\frac{5}{2}\ ,\ A_1=\frac{1}{2}\ ,\ A_2=\frac{1}{2}\ ,\ A_3=\frac{3}{2}$$

From (16.43) the area coordinates of p are

$$x_1=\frac{A_1}{A}=\frac{1}{5}\ ,\ x_2=\frac{A_2}{A}=\frac{1}{5}\ ,\ x_3=\frac{A_3}{A}=\frac{3}{5}$$

confirming that $\xi_3=1-\xi_1-\xi_2$. Substituting ξ_1, ξ_2 and ξ_3 into (16.84)

$$N_1 = (2x_1 - 1)x_1 = \left(2\frac{1}{5} - 1\right)\frac{1}{5} = -\frac{3}{25}$$

$$N_2 = (2x_2 - 1)x_2 = \left(2\frac{1}{5} - 1\right)\frac{1}{5} = -\frac{3}{25}$$

$$N_3 = (2x_3 - 1)x_3 = \left(2\frac{3}{5} - 1\right)\frac{1}{5} = \frac{3}{25}$$

$$N_4 = 4x_1 x_2 = 4\frac{1}{5}\cdot\frac{1}{5} = \frac{4}{25}$$

$$N_5 = 4x_2 x_3 = 4\frac{1}{5}\cdot\frac{3}{5} = \frac{12}{25}$$

$$N_4 = 4x_1 x_3 = 4\frac{1}{5}\cdot\frac{3}{5} = \frac{12}{25}$$

confirming that ΣN_i=1. The displacement (u,v) at point p is

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & \cdots & N_6 & 0 \\ 0 & N_1 & \cdots & 0 & N_6 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ \vdots \\ u_6 \\ v_6 \end{Bmatrix} = \begin{Bmatrix} 3.15 \\ 3.8 \end{Bmatrix}$$

16.6 From (16.60) the element shape functions are

$$N_1 = 2x^2 - 3x + 1 \ , \ N_2 = 2x^2 - x \ , \ N_3 = 4x - 4x^2$$

so that the derivatives with respect to ξ are

$$\frac{\P N_1}{\P x} = 4x - 3 \ , \ \frac{\P N_2}{\P x} = 4x - 1 \ , \ \frac{\P N_3}{\P x} = 4 - 8x$$

From (16.70) the Jacobian matrix is

$$[J] = \frac{\P x}{\P x} = \frac{\P N_1}{\P x}x_1 + \frac{\P N_2}{\P x}x_2 + \frac{\P N_3}{\P x}x_3 = (4x-3)1 + (4x-1)5 + (4-8x)3 = 4$$

16.7 From (16.53)

$$I_y = \iint_A x^2 dA = \int_0^1\int_0^1 f(x_1, x_2)|J|dx_1 dx_2 = 2A\sum_{i=1}^{n} w_i f(x_1, x_2)$$

noting that $|J|$=2A. With reference to Table 16.2 the x coordinate at the three integration points a, b and c are

$$x_a = \sum_{i=1}^{3} x_i^a x_i = 0\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 0\left(\frac{2}{3}\right) = \frac{1}{2}$$

$$x_b = \sum_{i=1}^{3} x_i^b x_i = 0\left(\frac{2}{3}\right) + 3\left(\frac{1}{6}\right) + 0\left(\frac{1}{6}\right) = \frac{1}{2}$$

$$x_c = \sum_{i=1}^{3} x_i^c x_i = 0\left(\frac{1}{6}\right) + 3\left(\frac{2}{3}\right) + 0\left(\frac{1}{6}\right) = 2$$

With A=6 and w_i=1/6 for all three integration points then I_y is

$$I_y = 2\text{x}6\left\{\frac{1}{6}\left(\frac{1}{2}\right)^2 + \frac{1}{6}\left(\frac{1}{2}\right)^2 + \frac{1}{6}2^2\right\} = 9$$

An exact evaluation of I_y can be performed by referring to Example 2.1. Considering an elemental strip dy at a distance y from the x-axis then

$$y = h - \frac{hx}{b}$$

With dA=ydx then I_y is

$$I_y = \iint_A x^2 dA = \int_0^b x^2 y dx = \int_0^b x^2 \left(h - \frac{hx}{b} \right) dx = \frac{h}{b} \int_0^b \left(bx^2 - x^3 \right) dx = \frac{hb^3}{12}$$

With b=3 and h=4 then I_y=4(3)^3/12=9 which agrees exactly with the evaluation of I_y using Gaussian integration.

Chapter 17 Solutions

17.1 Refer to §17.2 for a discussion of the stress intensity factor, §17.3 for a discussion of the *T*-stress and §17.5 for a discussion of several well-known stress intensity and *T*-stress expressions.

17.2 For a symmetrically cracked circular hole in a plate with far-field uniform loading consider the two limits of $a \to 0$ and $a \to \infty$ in the given expression. For $a \to 0$ we have

$$\left(\frac{R}{R+a}\right)^{2.4} \to 1 \quad \text{and} \quad K_I \to 3.365\sigma\sqrt{\pi a}$$

and for the limit $a \to \infty$ we have

$$\left(\frac{R}{R+a}\right)^{2.4} \to 0 \quad \text{and} \quad K_I \to \sigma\sqrt{\pi a}$$

Thus, as the cracks grow beyond the influence of the circular hole then K_I tends to the case of a centrally cracked plate.

The case of $a \to 0$ requires further analysis. When a crack is short ($a \ll R$) then the crack is approximately equivalent to an edge crack in a semi-infinite plate but with the applied stress modified by the circular hole stress concentration factor of 3σ. In this case K_I is given by

$$K_I = 1.1215(3\sigma)\sqrt{\pi a} = 3.3645\sigma\sqrt{\pi a}$$

which is essentially equivalent to the given equation by letting $a \to 0$. Similarly, for the *T*-stress as $a \to 0$ we have

$$T = -0.5258(3\sigma) = -1.5774\sigma$$

17.3 Refer to §17.8 for a discussion of plane strain fracture toughness and its experimental determination.

17.4 The total crack length is $2a$=25mm so that a=12.5mm. Neglecting finite plate effects with Y=1 then for case i) we have

$$K_{IC} = \sigma_f\sqrt{\pi a} = 220\sqrt{\pi\left(12.5\text{x}10^{-3}\right)} = 43.6\text{MNm}^{-3/2}$$

For case ii) with $\alpha = 1/\pi$ for Irwin's model then from (17.91) K_{IC} is

$$K_{IC} = \sigma_f\sqrt{\pi a\left[1 + \alpha\pi\left(\frac{\sigma_f}{\sigma_Y}\right)^2\right]} = 49.1\text{MNm}^{-3/2}$$

The discrepancy between the elastic and plasticity correction estimates increases as σ_f/σ_Y increases although the small-scale yielding assumption becomes increasingly invalid.

17.5 The rotor rotational speed in radians per second is

$$\omega = 12\text{x}10^3\left(\frac{2\pi}{60}\right) = 1{,}257\,\text{radians / second}$$

At the critical crack length then $K_I = K_{IC}$ and $a = a_{critical}$

$$K_{IC} = Yr\,\frac{w^2R^2}{8}\left(\frac{3-2n}{1-n}\right)\sqrt{pa_{critical}}$$

$$\Rightarrow\quad 85\text{x}10^6 = 0.55\left(8\text{x}10^3\right)\frac{1{,}257^2\left(0.35\right)^2}{8}\left(\frac{3-2\left(0.33\right)}{1-0.33}\right)\sqrt{pa_{critical}}$$

Solving for $a_{critical}$ we have

$$a_{critical} = 16.7\text{mm}$$

Comparing K_I with $\Delta K = Y\Delta s\sqrt{pa}$ then we observe that the cyclic stress is

$$\Delta s = r\,\frac{w^2R^2}{8}\left(\frac{3-2n}{1-n}\right) = 8\text{x}10^3\,\frac{1{,}257^2\left(0.35\right)^2}{8}\left(\frac{3-2\left(0.33\right)}{1-0.33}\right) = 675.5\text{x}10^6\,\text{MPa}$$

Since when operating the rotational speed is constant then the only way in which the stress cycles is from the starting-stopping of the rotor. Using this assumption then ΔN will provide us with the number of times that the rotor can be run up to speed. From (17.159) the number of cycles is

$$\Delta N = \frac{1}{C\Delta s^{m}p^{m/2}Y^{m}}\left[\frac{a_i^{1-m/2} - a_f^{1-m/2}}{m/2-1}\right] =$$

$$= \frac{1}{4.11\text{x}10^{-11}\left(675.5\right)^3 p^{3/2}\left(0.55\right)^3}\left[\frac{0.012^{1-3/2} - 0.017^{1-3/2}}{0.5}\right] = 250\text{ cycles}$$

17.6 Refer to §17.11 for a discussion on long and short fatigue cracks. From Hobson's growth law (17.164) with α=0 and re-arranging for the number of cycles we have

$$\Delta N = \frac{1}{C}\int_{a_i}^{a_f}\frac{da}{d-a} = -\frac{1}{C}\left[\ln\left(d-a\right)\right]_{a_i}^{a_f}$$

With $\Delta\sigma$=638MPa then C is found to be

$$C = 1.64\text{x}10^{-34}\left(638\right)^{11.14} = 2.887\text{x}10^{-3}\quad;\quad \frac{1}{C} = 346$$

With the long crack condition ΔK_{th}=6MPa√m and Y=2/π then from $\Delta K = Y\Delta s\sqrt{pa}$ the total threshold crack length is

$$a_{th} = 2\,\frac{\Delta K_{th}^2}{pY^2\Delta s^2} = \frac{2\left(6\right)^2}{p\left(\frac{2}{p}\right)^2\left(638\right)^2} = 138\text{x}10^{-6}m = 138mm \equiv d$$

With a_i=10μm and a_f=84% of d which is equal to 116μm then the number of cycles is

$$\Delta N = -346\left[\ln\left(138-116\right) - \ln\left(138-10\right)\right] = 609\text{ cycles}$$

17.7 For plane strain conditions then (κ+1)/8μ=(1-ν^2)/E=1/E'. Therefore, from (17.183) the mode I stress intensity factor is

$$K_I = \sqrt{E'J} = \sqrt{231\text{x}10^9\left(2.4\text{x}10^3\right)} = 23.55\text{MPa}\sqrt{\text{m}}$$

Chapter 18 Solutions

18.1 Refer to sections 18.1 to 18.8.

18.2 From (18.7) and taking logarithms we have

$$\ln \dot{e}_s = \ln A + n \ln s$$

From a $\ln \dot{e}_s$ versus lnσ plot with two 1 (σ=100MPa) and 2 (σ=200MPa) then the slope is

$$n = \frac{\ln \dot{e}_{s2} - \ln \dot{e}_{s1}}{\ln s_2 - \ln s_1} = \frac{\ln\left(7.6\text{x}10^{-4}\right) - \ln\left(4.4\text{x}10^{-6}\right)}{\ln 200 - \ln 100} = 7.4328$$

Assuming n to be an integer and equal to 7 then the constant A is, (18.7)

$$A = \frac{\dot{e}_s}{s^n} = \frac{4.4\text{x}10^{-6}}{100^7} = 4.4\text{x}10^{-20}$$

in which the first test point has been used.

18.3 From (18.10) and taking logarithms we have

$$\ln \dot{e}_s = \ln D - \frac{Q}{RT}$$

From a $\ln \dot{e}_s$ versus $1/T$ plot with two points 1 (T=160K) and 2 (T=200K) then the slope is

$$-\frac{Q}{R} = \frac{\ln \dot{e}_{s2} - \ln \dot{e}_{s1}}{1/T_2 - 1/T_1} = \frac{\ln\left(8.19\text{x}10^{-3}\right) - \ln\left(7.24\text{x}10^{-6}\right)}{1/200 - 1/160} = -5{,}625$$

from which Q is found to be

$$Q = 1.98\text{x}5{,}625 = 11\text{kcal/mol}$$

Finally, solving for D at T=160K we have

$$\ln D = \ln \dot{e}_s + \frac{Q}{RT} = \ln\left(7.24\text{x}10^{-6}\right) + \frac{11\text{x}10^3}{1.98(160)} \quad \Rightarrow \quad D = 8.7\text{x}10^9$$

18.4 For the equivalents strains in the two tests then

$$t_1 e^{-Q/RT_1} = t_2 e^{-Q/RT_2}$$

Taking logarithms we have

$$\ln t_1 + (-Q/RT_1) = \ln t_2 + \left(-Q/RT_2\right)$$

and re-arranging for Q we arrive at the required result

$$Q = \frac{RT_1T_2}{T_2 - T_1}\left[\ln t_1 - \ln t_2\right]$$

18.5 Re-arranging (18.29) for t/t_f we have

$$\frac{t}{t_f} = 1 - \left(\frac{A}{A_0}\right)^n = 1 - \left(\frac{3}{4}\right)^3 = 0.5781$$

18.6 From (18.43) the skeletal radius is

$$r_{sk} = \left(\frac{J}{J_c}\right)^{n/(n-1)} = \left[\frac{(1+3n)R^4}{4nR^{(1+3n)/n}}\right]^{n/(n-1)} = \left[\frac{(1+3\text{x}4)50^4}{4\text{x}4\text{x}50^{(1+3\text{x}4)/4}}\right]^{4/(4-1)} = 38\text{mm}$$

18.7 The elastic second moment of area for the rectangular beam is

$$I = \frac{b(2h)^3}{12} = \frac{50\text{x}75^3}{12} = 1{,}757{,}812\,\text{mm}^4$$

The creep second moment of area is, (18.47)

$$I_c = \frac{2bn}{2n+1} h^{(2n+1)/n} = \frac{2\text{x}50\text{x}5}{2\text{x}5+1}\left(\frac{75}{2}\right)^{(2\text{x}5+1)/5} = 131{,}961$$

Thus, the skeletal depth is given by, (18.52)

$$y_{sk} = \left(\frac{I}{I_c}\right)^{n/(n-1)} = \left(\frac{1{,}757{,}812}{131{,}961}\right)^{5/(5-1)} = 25.45\text{mm}$$

Chapter 19 Solutions

19.1 From (19.6) the strain after 100s using the Maxwell model is

$$\varepsilon = \left(\frac{t}{\mu} + \frac{1}{E}\right)\sigma_0 = \left(\frac{100}{1000\text{x}10^6} + \frac{1}{0.3\text{x}10^9}\right)5\text{x}10^3 = 517\text{x}10^{-6}$$

and according to the Voigt model, (9.16)

$$\varepsilon = \left(1 - e^{-Et/\mu}\right)\frac{\sigma_0}{E} = \left(1 - \exp\left[-100\text{x}10^3\text{x}100 / 10\text{x}10^6\right]\right)\frac{5\text{x}10^3}{100\text{x}10^3} = 32\text{x}10^{-6}$$

19.2 The total strain, ε, is the sum of the strains due to the Maxwell, ε_m, and Voigt, ε_v, models from (19.6) and (19.16)

$$\varepsilon = \varepsilon_m + \varepsilon_v = \left(\frac{t}{\mu_m} + \frac{1}{E_m}\right)\sigma_0 + \left(1 - e^{-E_v t/\mu_v}\right)\frac{\sigma_0}{E_v}$$

At t=0 the instantaneous elastic strain is $\varepsilon=\sigma_0/E_m$. Differentiating ε with respect to time then the steady creep rate is

$$\dot{\varepsilon} = \frac{\sigma_0}{\mu_m}\left[1 + \frac{\mu_m}{\mu_v}e^{-E_v t/\mu_v}\right]$$

As $t\to\infty$ then $\dot{\varepsilon} \to \sigma_0/\mu_m$. The variation of ε against t is shown in Figure Sol19.2. Clearly, when μ_m=0 then the model is equivalent to the standard linear solid model.

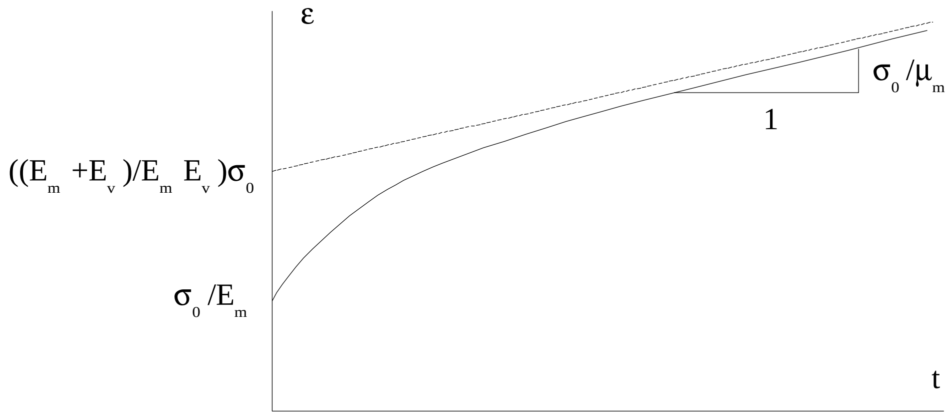


Figure Sol19.2. Creep response for the model of Exercise 19.2.

19.3 From the solution of Exercise 19.2 the strain is given by

$$\varepsilon = \varepsilon_m + \varepsilon_v = \left(\frac{t}{\mu_m} + \frac{1}{E_m}\right)\sigma_0 + \left(1 - e^{-E_v t/\mu_v}\right)\frac{\sigma_0}{E_v}$$

After the removal of stress σ_0 at time t=τ then the elastic strain σ_0/E is instantaneously recovered followed by the recovery strain of, from (19.6) and (19.22)

$$\varepsilon = \frac{\sigma_0 t'}{\mu_m} + \frac{\sigma_0}{E_v}\left(1 - e^{-E_v t/\mu_v}\right)e^{-E_v t'/\mu_v}$$

As $t\to\infty$ then $\varepsilon\to\sigma_0 t'/\mu_m$. The strain-time curve is illustrated in Figure Sol19.3.

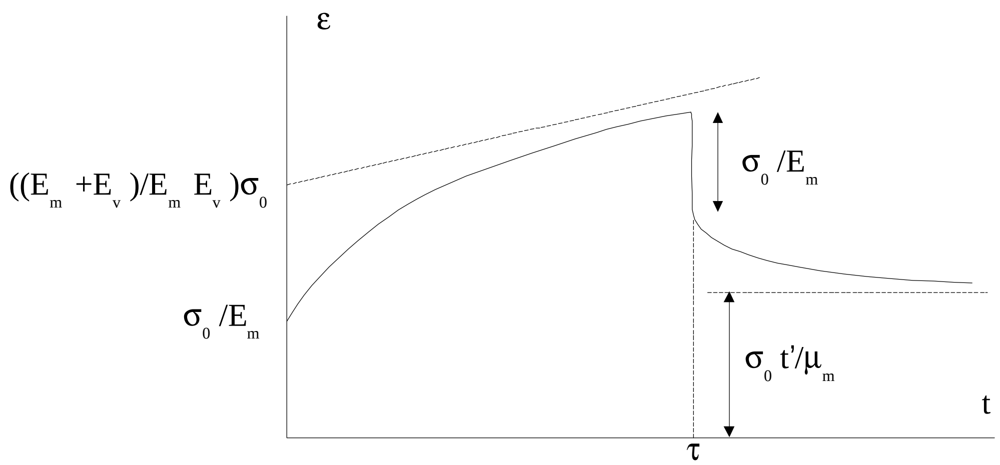


Figure Sol19.3. Strain-time curve for the model of Exercise 19.2 with removal of stress σ_0 at time t=τ.

19.4 From (9.33) the relaxation stress at time t=10s is

$$s\left(10s\right) = \frac{0.8\text{x}10^9(10\text{x}10^{-6})}{(0.8+0.2)\text{x}10^9}\left[0.2\text{x}10^9 + 0.8\text{x}10^9 \exp[-(0.8+0.2)\text{x}10^9\,\text{x}10/5\text{x}10^9\right] = 2.466\text{kPa}$$

19.5 From (19.6) and (19.10) the creep compliance, $C(t)$, and relaxation modulus, $G(t)$, for the Maxwell model are

$$C(t) = \frac{t}{m} + \frac{1}{E} \quad , \quad G(t) = Ee^{-Et/m}$$

and from (19.16) the creep compliance function for the Voigt model is

$$C(t) = \frac{1-e^{-Et/m}}{E}$$

with $G(t)$=0.

19.6 Following a similar procedure as used in the derivation of (19.6) then the total strain rate is

$$\dot{e} = \dot{e}_s + \dot{e}_d = \frac{1}{E}\dot{s} + As^n$$

For constant stress σ=σ_0 then

$$\dot{e} = As_0^n \quad \text{or} \quad e = As_0^n + B$$

where B is a constant of integration. If at time t=0 the instantaneous elastic strain is σ_0/E then $B=\sigma_0/E$ and ε is given by

$$e = \frac{s_0}{E} + As_o^n t$$

19.7 From (19.42) the strain at time t=3500s is

$$e(3500) = 0.5x10^6(1.52\text{x}10^{-9})(3500)^{0.15} +$$
$$0.25\text{x}10^6(1.52\text{x}10^{-9})(3500-1000)^{0.15} +$$
$$0.25\text{x}10^6(1.52\,\text{x}10^{-9})(3500-2000)^{0.15} -$$
$$1\text{x}10^6\,\text{x}(1.52\text{x}10^{-9})(3500-3000) = 1.15\text{x}10^{-3}$$

Chapter 20 Solutions

20.1 Refer to §20.2 for a summary of different types of damage.

20.2 From (20.1) the damage parameter is

$$\omega = \frac{A}{A_0} = \frac{826}{1{,}018} = 0.81$$

20.3 From (20.2) the continuity parameter is

$$\psi = 1 - \omega = 1 - 0.81 = 0.19$$

20.4 From (20.2) and (20.9)

$$\omega = 1 - \psi = 1 - \frac{E'}{E} = 1 - \frac{67}{190} = 0.65$$

20.5 From (20.15)

$$\sigma_0^{-n} = A(n+1)t_f = Bt_f$$

Letting *m*=-*n* and taking natural logarithms we have

$$\ln t_f = m \ln \sigma_0 - \ln B$$

From a ($\ln\sigma_0$, $\ln t_f$) plot then *m* is given by, using the first and last data points

$$m = \frac{\ln 2.17 - \ln 97.11}{\ln 14 - \ln 10} = -13.32 \quad ; \quad n = -m = 13.32$$

and with *B* given by

$$\ln B = -(n \ln \sigma_0 + \ln t_f) = -(13.32 \ln 14 + \ln 2.17) = -35.9269$$

from which *B* is found

$$B = \exp(-35.9269) = 2.5\text{x}10^{-16}$$

The constant *A* now follows

$$A = \frac{B}{n+1} = \frac{2.5\text{x}10^{-16}}{13.32+1} = 1.75\text{x}10^{-17}$$

20.6 From (20.17) ψ is given by

$$\psi = (1-0.8)^{1/(13.32+1)} = 0.8937$$

and ω is equal to ω=1-ψ=1-0.8937=0.1063.

20.7 From (20.30) λ is

$$\lambda = \frac{\varepsilon_f}{A\sigma^n t_f}$$

At the start of tertiary creep *t*=0 and ω=0 then from (20.23)

$$\dot{\varepsilon}_s = A\sigma^n$$

Substituting $\dot{\varepsilon}_s$ then λ is given by

$$\lambda = \frac{\varepsilon_f}{\dot{\varepsilon}_s t_f}$$

as required.

Chapter 21 Solutions

21.1 From (21.1) and (21.3) the maximum volume fractions (r=R) for square and hexagonal fibre configurations are

$$V_{f,\max}(sq) = \frac{p}{4}\left(\frac{r}{R}\right)^2 = \frac{p}{4} \approx 0.785$$

$$V_{f,\max}(hex) = \frac{p}{2\sqrt{3}}\left(\frac{r}{R}\right)^2 = \frac{p}{2\sqrt{3}} \approx 0.907$$

From (21.13) and (21.18) $E_{\parallel}$ and $E_{\perp}$ for the square and hexagonal configurations at $V_{f,max}$ are found to be

$$\frac{E_{\parallel}(hex,V_{f,\max})}{E_{\parallel}(sq,V_{f,\max})} = \frac{0.907+0.093n}{0.785+0.215n} \quad , \quad \frac{E_{\perp}(hex,V_{f,\max})}{E_{\perp}(sq,V_{f,\max})} = \frac{0.215+0.785n}{0.093+0.907n}$$

where n=E_m/E_f. For example, when n=0.1 then $E_{\parallel}(hex,V_{f,max})/E_{\parallel}(sq,V_{f,max})$=1.14 and $E_{\perp}(hex,V_{f,max})/E_{\perp}(sq,V_{f,max})$=1.6.

21.2 From the rule of mixtures (21.13)

$$E_{\parallel} = E_f V_f + E_m\left(1-V_f\right) = 76(0.45) + 4(1-0.45) = 36.4\text{GPa}$$

21.3 From (21.19)

$$E_{\perp} = \frac{E_f E_m}{E_f\left(1-V_f\right)+E_m E_f} = \frac{76\text{x}4}{76(1-0.45)+4(0.45)} = 6.97\text{GPa}$$

21.4 With E_1=$E_{\parallel}$=36.4GPa from Exercise 21.2 and E_2=$E_{\perp}$=6.97GPa from Exercise 21.3 then $\overline{E}$, $\overline{G}$ and $\overline{n}$ from (21.34) are

$$\overline{E} = \frac{3}{8}E_{\parallel} + \frac{5}{8}E_{\perp} = \frac{3}{8}(36.4) + \frac{5}{8}(6.97) = 18\text{GPa}$$

$$\overline{G} = \frac{1}{8}E_{\parallel} + \frac{1}{4}E_{\perp} = \frac{1}{8}(36.4) + \frac{1}{4}(6.97) = 6.29\text{GPa}$$

$$\overline{n} = \frac{\overline{\text{E}}}{2\overline{\text{G}}} - 1 = \frac{18}{2(6.29)} - 1 = 0.43$$

21.5 From (21.38) the volume fraction at cross-over between low and high V_f is

$$V_f^{'} = \frac{s_m^*}{s_f^* - s_f^{'} + s_m^*} = \frac{75}{2{,}000 - 1{,}500 + 75} = 0.13$$

If $V_f > V_f^{'}$ then fibre strength dominates.

21.6 From (21.43) the transverse failure strength, $s_{\perp}^*$, is

$$s_{\perp}^* = s_m^*\left(1 - 2\sqrt{\frac{V_f}{p}}\right) = 65\left(1 - 2\sqrt{\frac{0.3}{p}}\right) = 24.83\text{MPa}$$

21.7 From (21.45) the critical embedded fibre length is

$$l_{ce} = \frac{s^{*}_{f} r}{2t} = \frac{750\text{x}10^{6}\left(0.2\text{x}10^{-3}\right)}{2\left(45\text{x}10^{6}\right)} = 1.7\,\text{mm}$$

Chapter 22 Solutions

22.1 Resolving P_r we find that $P=P_r\sin\theta$ and $Q=P_r\cos\theta$ and upon substitution into (22.7) and (22.18) and superimposing the stresses gives the stresses in the half-plane.

22.2 The stresses for a point force at $x=b$ can be found by replacing x by (x-b) in (22.7)

$$s_{xx} = -\frac{2P(x-b)^2 y}{p\left[(x-b)^2 + y^2\right]^2}$$

$$s_{yy} = -\frac{2Py^3}{p\left[(x-b)^2 + y^2\right]^2}$$

$$t_{xy} = -\frac{2P(x-b)y^2}{p\left[(x-b)^2 + y^2\right]^2}$$

To show that τ_{xy} is zero along the line $x=b/2$ write expressions for τ_{xy} when P is at $x=0$ and $x=b$, from (22.7) and the above equation

$$t_{xy}(P \text{ at } x=0) = -\frac{Pby^2}{p\left[(b/2)^2 + y^2\right]^2} \quad , \quad t_{xy}(P \text{ at } x=b) = \frac{Pby^2}{p\left[(b/2)^2 + y^2\right]}$$

which cancel if superimposed.

22.3 Referring to Figure 22.26 then the applied pressure distribution is given by

$$p(x) = \frac{p_0(a-|x|)}{a} \quad ; \quad |x| \le a$$

Performing the integrations in (22.20) then the stresses are found to be

$$s_{xx} = \frac{p_0}{pa}\left\{(x-a)q_1 + (x+a)q_2 - 2xq + 2y\ln\left(\frac{r_1 r_2}{r^2}\right)\right\}$$

$$s_{yy} = \frac{p_0}{pa}\left\{(x-a)q_1 + (x+a)q_2 - 2xq\right\}$$

$$t_{xy} = -\frac{p_0 y}{pa}(q_1 + q_2 - 2q)$$

where r, r_1, r_2, θ, θ_1 and θ_2 are given by

$$r = x^2 + y^2 \quad , \quad r_1^2 = (x-a)^2 + y^2 \quad , \quad r_2^2 = (x+a)^2 + y^2$$

$$\tan q = \frac{y}{x} \quad , \quad \tan q_1 = \frac{y}{x-a} \quad , \quad \tan q_2 = \frac{y}{x+a}$$

22.4 For a uniform pressure ($N=P$ and $T=0$) then $\Phi(z)$ and $\Psi(z)$ are

$$\Phi(z) = -\frac{p}{2pi}\int_{-a}^{a}\frac{dt}{z-t} = \frac{p}{2pi}\left[\ln(z-t)\right]_{t=-a}^{t=a} = \frac{p}{2pi}\ln\left(\frac{z-a}{z+a}\right)$$

$$\Psi(z) = -\frac{zp}{2pi}\int_{-a}^{a}\frac{dt}{(t-z)^2} = -\frac{paz}{pi\left(z^2 - a^2\right)}$$

With reference to Figure Sol22.4 then let $z\text{-}t=re^{-i\theta}$ where $r=|z\text{-}t|$. From (14.148) and (14.152) the stress components are

$$s_{xx}+s_{yy}=4\,\mathrm{Re}\,\Phi(z)=-\frac{2p}{p}(q_1-q_2)$$

$$s_{yy}-s_{xx}+2it_{xy}=2\left[\bar{z}\Phi'(z)+\Psi(z)\right]=\frac{2pa}{pi}\left(\frac{\bar{z}-z}{z^2-a^a}\right)=-\frac{4pay}{p\left(z^2-a^2\right)}$$

Solving for σ_{xx}, σ_{yy} and τ_{xy} we arrive at

$$s_{xx}=-\frac{p}{p}(q_1-q_2)+\frac{2pay\left(x^2-y^2-a^2\right)}{p\left[\left(x^2+y^2-a^2\right)^2+4a^2y^2\right]}$$

$$s_{yy}=-\frac{p}{p}(q_1-q_2)-\frac{2pay\left(x^2-y^2-a^2\right)}{p\left[\left(x^2+y^2-a^2\right)^2+4a^2y^2\right]}$$

$$t_{xy}=\frac{4paxy^2}{p\left[\left(x^2+y^2-a^2\right)^2+4a^2y^2\right]}$$

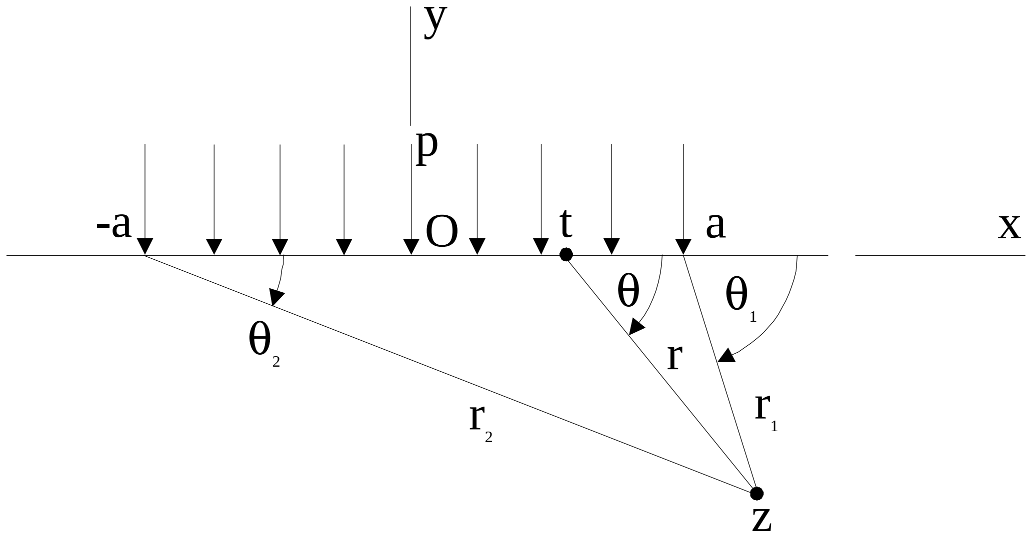


Figure Sol22.4. A half-plane subject to a uniform normal pressure.

22.5 From (22.55) the pressure beneath the centre of the punch is

$$p(0)=\frac{P}{pa}=\frac{5\text{x}10^3}{p\left(12.5\text{x}10^{-3}\right)}=0.127\,\text{MPa}$$

22.6 From (22.70) the radius of the contact radius, *a*, and total displacement, δ, are

$$a=\left[\frac{3P}{4}\frac{R_1R_2}{R_1+R_2}\left(\frac{1-n_1^2}{E_1}+\frac{1-n_2^2}{E_2}\right)\right]^{1/3}$$

$$d=\left[\frac{9P^2}{16}\frac{R_1+R_2}{R_1R_2}\left(\frac{1-n_1^2}{E_1}+\frac{1-n_2^2}{E_2}\right)^2\right]^{1/3}$$

Letting body 2 be the half-plane, R_1=∞, and R=R_2 then $R_1R_2/(R_1+R_2)$ is found to reduce to the following

$$\frac{R_1 R_2}{R_1 + R_2} = \frac{R_1 R_2}{R_1\left(1 + R_2 \ / \ R_1\right)} = R_2 = R \quad \text{as} \quad R_1 \to \infty$$

In addition, letting E=E_1=E_2 and ν=ν_1=ν_2 for both bodies having the same elastic properties then the above expressions for a and δ are found to reduce to

$$a = \left[\frac{3PR}{2}\left(\frac{1-n^2}{E}\right)\right]^{1/3} \quad , \quad d = \left[\frac{9P^2}{16R}\left(\frac{1-n^2}{E}\right)^2\right]^{1/3}$$

as required.

22.7 We can obtain expressions for a, δ and p_0 for the circular ball in a circular seat from (22.70) by simply making R_1 negative.

Chapter 23 Solutions

23.1 Table Sol23.1 summarises the failure strengths of the tested composite material.

failure strength	ab. frequency	rel. frequency	cum. Ab. frequency	cum. rel. frequency
650	1	0.1	1	0.1
680	2	0.2	3	0.3
700	2	0.2	5	0.5
710	3	0.3	8	0.8
740	1	0.1	9	0.9
750	1	0.1	10	1

Table Sol23.1. Sample of 10 values of the tensile failure strength (MPa) of glass-polyester unidirectional laminae composite specimens.

23.2 From (23.4) the mean is

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i = 703\text{MPa}$$

From (23.5) the variance is

$$s^2 = \frac{1}{N-1}\sum_{i=1}^{N}\left(x_i - \bar{x}\right)^2 = 845.55\text{MPa}$$

and the standard deviation is the square root of the variance

$$s = \sqrt{s^2} = 29.08\text{MPa}$$

23.3 Since a thrown die can result in a number which is both even and a multiple of 3 then both events can occur simultaneously and therefore the events are arbitrary. Letting A represent an even number and B represent a number which is a multiple of 3 then A={2,4,6} and B={3,6}. The required probabilities for arbitrary events are $P(A)$=3/6=1/2, $P(B)$=2/6=1/3 and $P(A\cap B)$=1/6 so that from (23.13) we have

$$P(A\cup B) = P(A) + P(B) - P(A\cap B) = \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3}$$

23.4 From (23.28) the mean is

$$\bar{x} = \int_a^b xf(x)dx = \int_0^3 x\left(\frac{x}{2}\right)dx = \left[\frac{x^3}{6}\right]_0^3 = \frac{9}{2}$$

From (23.30) the variance is

$$s^2 = \int_a^b \left(x-\bar{x}\right)^2 f(x)dx = \int \left(x - \frac{9}{2}\right)^2\left(\frac{x}{2}\right)dx = \left[\frac{x^4}{8} - \frac{3x^3}{2} + \frac{81x^2}{16}\right]_0^3 = \frac{243}{16}$$

23.5 For case i) then $\bar{x}$ =1 and σ^2=1 so that we can obtain the probability directly from Table 23.2, and is found to 0.9772. For case ii) with $\bar{x}$ =0.4 and σ^2=4 then the standardised variable is

$$z = \frac{x-\bar{x}}{s} = \frac{2-0.4}{2} = 0.8$$

From Table 23.2 we have $\Phi(0.8)$=0.7881.

23.6 From (23.53) and Table 23.3 the mean strength is

$$\bar{s} = \frac{s_0}{V^{1/m}}\Gamma\left(1+\frac{1}{m}\right) = \frac{650}{0.225^{1/10}}\Gamma\left(1+\frac{1}{10}\right) = 718\text{MPa}$$

23.7 Letting the specimen of Exercise 23.6 be denoted by specimen 1 and the new specimen by 2 then from (23.65) the expected mean strength of the new specimen is, with $\bar{s}_1$=718MPa

$$\bar{s}_2 = \frac{\bar{s}_1}{(V_2 / V_1)^{1/m}} = \frac{718}{(0.45 / 0.225)^{1/10}} = 670\text{MPa}$$